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Observation boundaries for destructive quantum readout and external geometry

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Abstract

We specify an observation boundary for a finite linearly typed quantum interface with destructive computational-basis readout and primitive classical geometric labels. Geometric rules inspect declared labels; coupling rules inspect declared matter observables; control and resource checks are part of the interface. Under spectator-blind control, arbitrary extensions with the same accessible reduced state preserve geometric traces and accessible matter states. A restricted coherent implementation of destructive readout agrees with immediate readout on their common record–matter output. For a channel T, coarse effects E_w, and aggregated fine effects Π_w, record compatibility is equivalent to T^{*}(E_w) = Π_w. The exact unassisted worst-case total-variation error is a maximum of operator norms over outcome events. Exact record compatibility extends to untouched entangled references, but does not determine residual instruments. For sharp measurements and a coarse Lüders instrument, full compatibility is characterized exactly by the removal of fine coherence within each coarse outcome. A compositional criterion identifies when continuations preserve agreement on a declared observable algebra. One complete four-face adjacency exchange illustrates the interface. It is induced by a prescribed slicing of a fixed conical spacetime and has explicit charts, edge laws, a compact collar, a transition stratum, an exact guard, location transport, and a decoder-correct finite reversible implementation. The general interface results are independent of this example. A separate dictionary proves three linear observation rules for distributional comparisons without assuming any particular renormalized composite.

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The closed observation problem

Destructive measurement, a coherent record carrier, and a measurement instrument with a surviving quantum output are different interfaces. A geometric extension of a quantum language must say which of these interfaces it can observe. Linear ownership of quantum names alone does not answer that question.

We take destructive computational-basis readout as a given primitive, with type

$$\mathrm{Enter} : \mathrm{Q}^{\otimes n} \to \mathrm{Bit}^{\otimes n}.$$

Its measured quantum interface is consumed. We use its Born semantics, as in the corresponding finite circuit fragment of the Enter calculus [Cheung2026Enter]. No characterization or uniqueness proof for the measurement

law is repeated here.

The semantic additions are fixed throughout: an external classical geometric configuration, locations for active names, declared coupling algebras, and an explicit control and resource interface. The question is then precise: Which spectator extensions, deferred implementations, and scale/readout comparisons preserve the outputs that the interface declares accessible? Our answers concern equality of observable outputs, not a mechanism that constructs geometry from quantum syntax.

Results and dependencies

The paper proves six interface statements.

• On a prescribed substrate, block channels have an exact finite-step backwards support bound. Retiming a fixed labelled gate partial order preserves the total channel.

• Under spectator-blind control and resources, any admissible spectator extension with the same accessible reduction preserves the geometric trace and that reduction at every step.

• Immediate destructive readout agrees with a restricted coherent-then-readout implementation after both outputs are restricted to the same record–matter interface.

• Record laws across evolution and label aggregation agree exactly when their effects obey a pullback identity. Their unassisted worst-case total-variation error has an exact event-norm formula.

• Record agreement does not imply residual-instrument agreement. For sharp measurements with a coarse Lüders instrument, an exact criterion isolates the remaining intra-outcome coherence.

• Agreement on an input observable algebra is preserved as agreement on an output observable algebra exactly when the adjoint continuation maps the latter into the former. A corresponding statement is given for classical block-diagonal state spaces.

Finite-dimensional channel conventions follow Watrous [Watrous2018]. The circuit principle of deferred measurement has a precise treatment in Gurevich and Blass [GurevichBlass2022]. Our emphasis is the additional observation boundary: geometry, control, resource tests, and accessible residual observables are all named.

The geometric certificate uses a fixed conical spacetime and the polygonal-slicing setting of [DeserJackiwHooft1984,tHooft1992,tHooft1993,Kadar2005]. The proofs of the interface theorems do not depend on that certificate. The certificate shows that one nontrivial change of cell incidence can meet the declared interface; it does not supply the assumptions of the interface theorems.

Operational ambition and limits

Within the proposed model, a future gravitational update may inspect only its declared observable interface. Destroyed names, inaccessible carriers, and resource bookkeeping are not undeclared geometric source variables. This is an operational requirement for a later quantum–geometric composite, not a derivation of its dynamics.

In particular, the paper proves no gravitational field equation, continuum limit, curvature-convergence theorem, physical causal dominance, or interacting-product theorem. It does not evaluate a singular triangle graph or establish a distributional finite part for one. The short distributional dictionary in contains conditional linear inference rules only.

The distinction is structural:

names are not mass, capacity is not curvature, readout is not coarse-graining.

Likewise, a scheduled circuit past is not a physical light cone. A computational implementation requires an additional physical model before such identifications can be tested.

Types, states, and external geometry

Finite-dimensional conventions

For a finite-dimensional Hilbert space $\mathscr{H}$, write $L(\mathscr{H})$ for its operators and $D(\mathscr{H})$ for its density operators. A channel is a completely positive, trace-preserving linear map. Its adjoint satisfies

$$\operatorname{Tr}(A\Phi(X)) = \operatorname{Tr}(\Phi^{*}(A)X).$$

The adjoint of a channel is unital and completely positive. A Kraus representation is

$$\Phi(X) = \sum_{\alpha} K_{\alpha} X K_{\alpha}^{\dagger}, \qquad \sum_{\alpha} K_{\alpha}^{\dagger} K_{\alpha} = I.$$

These are the standard finite-dimensional conventions [Watrous2018].

A classical register has a distinguished orthonormal basis. Admissible joint states are block diagonal in that basis, including their correlations with other registers. A diagonal classical marginal alone is insufficient. Operations declared to preserve classicality preserve this joint block-diagonal state space.

A unital observable algebra means a unital ^*-subalgebra $\mathcal{A} \subseteq L(\mathcal{H})$. Its accessible state functional is

$$\omega_{\rho}|_{\mathcal{A}} : A \text{longmapsto} \operatorname{Tr}(\rho A).$$

Agreement on $\mathcal{A}$ means equality of these functionals.

Our norm and distance conventions are

$$\|X\|_1 = \operatorname{Tr}\sqrt{X^{\dagger}X}, \qquad d_{\mathrm{tr}}(\rho,\sigma) = \frac{1}{2}\|\rho - \sigma\|_1, \qquad d_{\mathrm{TV}}(p,q) = \frac{1}{2}\sum_{w} |p(w) - q(w)|.$$

The operator norm is ‖·‖_{∞}. No ancillary reference is included in an optimization unless explicitly stated.

A minimal linear interface

Wire types are generated by

$$A ::= I \mid \mathrm{Q} \mid \mathrm{Bit} \mid A \otimes A.$$

A judgement $f: A \to B$ consumes the complete input interface and returns the declared output interface. Composition and tensoring obey

$$\frac{f : A \to B \quad g : B \to C}{g \circ f : A \to C}, \qquad \frac{f : A \to B \quad g : C \to D}{f \otimes g : A \otimes C \to B \otimes D}.$$

Identities and wire permutations are available. Circuit constants used below include quantum unitaries, preparation of $|0\rangle$, and destructive computational-basis readout. The finite classical-code implementation also uses reversible NOT, CNOT, and Toffoli operations on classical interfaces. Their quantum-carrier realizations are the corresponding permutation unitaries.

Linearity is an ownership condition. It permits a sequence of gates to consume and return the same physical tensor factor. It provides neither duplication of an arbitrary quantum wire nor an implicit discard operation.

The semantic matter-rule class below permits declared channels. We do not claim that every such channel has an exact implementation in the displayed finite gate library. Exact gate realizability is proved only for the finite reversible event code in .

Three measurement interfaces

Let S have computational basis $\{|z\rangle : z \in Z\}$, and let B be its unmeasured complement. Put

$$M_z = \langle z|_S \otimes I_B, \qquad P_z = |z\rangle\langle z|_S \otimes I_B.$$

Definition. The interpretation of *Enter_S* is the channel

$$\mathcal{M}_S(\rho) = \sum_{z} |z\rangle\langle z|_R \otimes M_z \rho M_z^{\dagger}, \qquad \mathcal{M}_S : L(\mathcal{H}_{SB}) \to L(\mathcal{H}_{RB}).$$

The record R is classical. The factor S is absent from the output.

Definition. A fresh quantum carrier A, with basis indexed by Z, is introduced by the isometry

$$V(|z\rangle_S \otimes |b\rangle_B) = |z\rangle_S \otimes |z\rangle_A \otimes |b\rangle_B.$$

The corresponding channel is $\mathit{Copy_S}(\rho) = V\rho V^{\dagger}$. It returns both S and A.

Definition. The represented computational-basis instrument is

$$\mathcal{I}_S^{\mathrm{nd}}(\rho) = \sum_{z} |z\rangle\langle z|_R \otimes P_z \rho P_z.$$

Its residual output is on $S \otimes B$. It is not the destructive primitive.

toprule Operation Measured factor Record interface Status

midrule *Enter_S S* consumed Classical *R* Primitive

Copy_S S retained Quantum carrier *A* Isometry

𝒥_S^nd S retained Classical *R* Represented instrument

bottomrule

Coherent recording is not cloning. For a qubit,

$$V|+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \neq |+\rangle \otimes |+\rangle.$$

The three interfaces must therefore remain distinct both in theorem statements and in geometric couplings.

Geometric configurations and admissibility

Definition. An external geometric interface supplies:

• a set $\mathcal{Q}$ of classical geometric and declared resource labels;

• an admissible subset

$$\mathcal{Q}_{\mathrm{adm}} = \mathcal{Q}_{\mathrm{reg}} \cup \mathcal{Q}_{\mathrm{tr}};$$

• a finite cell-label set *Cell (g)* for each admissible *g*;

• support, boundary, and resource certificates for permitted updates;

• an explicitly supplied gauge equivalence, when one is used. Regular configurations satisfy the model's nondegeneracy conditions. A transition configuration is admitted only with an event certificate specifying its realization and its permitted one-sided continuations.

In the example, positive-length edges belong to regular cellulations. At the event, a vanishing edge slot and coincident endpoint labels may remain in a germ description. Their realization is a single junction; they are not regular embedded one-cells.

Definition. A configuration is

$$C = (g; \Gamma; \rho; \mathrm{loc}; h), \qquad \mathrm{loc} : \Gamma \to \mathrm{Cell}\,(g),$$

where $g \in \mathcal{Q}_adm$, Γ is a finite labelled matter interface, $\rho \in D(\mathcal{H}_\{\Gamma\})$, and *h* is a finite control state. Classical names satisfy the joint classicality convention. The geometric projection is $\pi_g(C) = g$.

A location is an abstract cell label, not an undeclared developed position or a stress-energy assignment. Input and output matter interfaces may change. For example, destructive readout replaces measured names with fresh classical record names.

The three rule kinds

Definition. A finite execution uses three kinds of certified rule.

• A *geometric rule*, or *G*-rule, changes supported geometric labels as a specified function of geometric and declared resource labels. It does not inspect the matter density operator or matter-name cardinality. It returns a complete admissible configuration of geometric labels and a location transport map.

A transition rule includes its common event realization and both one-sided continuations.

• A *matter rule* applies a declared channel

$$\Phi_g : L(\mathcal{H}_{\Gamma_0}) \to L(\mathcal{H}_{\Gamma_0'})$$

on a declared interface, extended by the identity on unacted-on names. It may depend on the current geometric labels but does not change them. Its output-name locations are declared.

• A *coupling rule* declares an accessible unital algebra $\mathcal{A}_v$ and updates geometry by

$$g' = F_v(g, \omega_\rho|_{\mathcal{A}_v}).$$

It leaves the matter density operator unchanged and supplies a complete admissibility certificate and location transport.

A coupling update must return all affected labels needed for admissibility. Changing a particle parameter while silently freezing incompatible gluing data is not a certified update.

The control state, guards, and resource checks determine which rule is selected. Our executions are nonselective density-operator executions. A selectively normalized measurement branch is not silently treated as a trace-preserving matter rule. Classical control can instead be represented inside a channel that retains its records.

Assumption. For the compared executions, equal geometric labels, equal finite control states, and equal declared accessible state functionals select the same:

• rule and matter channel;

• applicability and capacity-check result;

• support and resource certificate;

• transport of original names and successor control state. Adding inaccessible spectator names does not change these selections. Spectator storage and required transport history are reserved separately from the geometric-update budget. Random external control is compared under a common seed or an explicitly coupled control law.

This is a hypothesis about the complete controller, not merely about the written coupling formula.

Remark. The map in need not be affine in ρ. Consequently the complete coupled input–output map need not be a quantum channel. The proofs below compare equal accessible inputs step by step. They do not establish general no-signalling or a physical consistency theorem for arbitrary nonlinear backreaction.

Gauge and observable geometry

A gauge equivalence is an explicitly supplied equivalence relation on descriptions. An admissible geometric observable is constant on the specified classes. Chart changes and retriangulations may be declared gauge when they represent the same metric and marked slice.

Equality of an ambient spacetime does not imply equality of every observable of different marked slices. Likewise, equality of total channels under a prescribed circuit retiming is not automatically a gauge symmetry of coupled geometric evolution.

Prescribed kinematics and schedule dependence

All geometry, partitions, and channel lists in this section are prescribed. No state-dependent controller changes them.

Exact backwards support

Let (X, d) be a finite metric space, with site Hilbert spaces $\mathscr{H}_x$. For $Y \subseteq X$, set

$$\mathscr{H}_X = \operatorname{bigotimes}_{x \in X} \mathscr{H}_x, \qquad \mathscr{A}(Y) = L\big(\operatorname{bigotimes}_{x \in Y} \mathscr{H}_x\big) \otimes I_{X \setminus Y}.$$

At tick t, let $\mathscr{B}_t$ be a partition of X, and suppose

$$\operatorname{diam} B \le \Delta, \qquad \Phi_t = \operatorname{bigotimes}_{B \in \mathscr{B}_t} \Phi_{t,B},$$

where every block map is a channel. If ticks have duration $\tau > 0$, the resulting scalar schedule bound is Δ/τ.

Define the backwards sets recursively by

$$\begin{aligned} N_{t,t}(Y) &= Y, \\ N_{s,t}(Y) &= \bigcup \{B \in \mathscr{B}_{s+1} : B \cap N_{s+1,t}(Y) \neq \varnothing\}, \qquad s < t. \end{aligned}$$

Lemma. For a block channel $\Phi = \operatorname{bigotimes}_{B \in \mathscr{B}} \Phi_B$,

$$\Phi^*(\mathscr{A}(Y)) \subseteq \mathscr{A}\Big(\bigcup_{B \cap Y \neq \varnothing} B\Big).$$

Proof. The complement of the displayed union is a union of blocks on which the input observable is the identity. Each corresponding adjoint maps the identity to itself. The other blocks can enlarge support only within their own union.

Theorem. For $A \in \mathscr{A}(Y)$ and $0 \le s \le t$,

$$\operatorname{supp}\left((\Phi^{*}_{s+1} \circ \cdots \circ \Phi^{*}_{t})(A)\right) \subseteq N_{s,t}(Y),$$

and

$$N_{s,t}(Y) \subseteq \{x : d(x, Y) \le (t - s)\Delta\}.$$

The output reduction on Y depends on the state at tick s only through its reduction to $N_{s, t}(Y)$.

Proof. Repeated application of in reverse tick order proves .

If $x \in N_{s, t}(Y)$, its block meets $N_{s + 1, t}(Y)$ at some y. Hence

$$d(x, Y) \le d(x, y) + d(y, Y) \le \Delta + (t - s - 1)\Delta.$$

Induction proves .

Finally, every output observable on Y pulls back to an observable on $N_{s, t}(Y)$. Equal input reductions there give equal expectations for all output observables, and therefore equal output reductions.

Corollary. A prescribed readout on Y at tick t depends only on the initial reduction to $N_{0, t}(Y)$. Replacing a prescribed block channel outside its backwards dependency region does not change its law, provided the supports and schedule remain fixed.

Proof. Apply to the readout effects. A replacement outside the backwards region acts on an identity factor in the pulled-back effect, and every channel adjoint preserves that identity.

Readout can be represented on fixed site spaces by reserving output record registers. An inactive placeholder after destructive measurement is not a surviving accessible measured factor.

The backwards region is an exact finite-step upper bound, not necessarily the smallest support for a particular list of channels. Joint record laws require the backwards region of the complete record-retaining network.

A relay attaining the scalar bound

Proposition. Let $R \ge 1$ be an integer. On $\mathbb{Z}$, with $d(i, j) = a|i - j|$, use the blocks

$$B_{t,k} = \{k(R + 1) + (t - 1)R + j : 0 \le j \le R\}.$$

Swap the endpoint factors of every block at each tick. A factor initially at 0 is at tR after t ticks, attaining speed Ra/τ. In units $R = a = \tau = 1$, the attained speed is 1.

Proof. At each tick the blocks are disjoint consecutive intervals of diameter Ra. At tick t, the transported factor is at the left endpoint $(t - 1)R$ of $B_{t, 0}$. The endpoint swap moves it to tR, which is the left endpoint of $B_{t + 1, 0}$. Induction proves the claim.

For any finite horizon, retain a finite interval containing all blocks used by this relay. Keep complete blocks inside that interval and complete cut boundary fragments with singleton blocks. No infinite tensor-product state is needed.

The transported factor may be entangled with an untouched reference; the argument is an identity of wire permutations.

Retiming fixed labelled channels

Definition. A finite collection of fixed labelled channels has declared supports $S_g \subseteq X$. A strict partial order is admissible if channels with intersecting supports are comparable. A legal execution is a linear extension of this order.

Theorem. All legal executions of an admissibly ordered labelled channel collection have the same total channel.

Proof. Incomparable channels have disjoint supports, so their extensions by identities commute. Any two linear extensions are connected by adjacent swaps of incomparable elements.

For completeness, compare two extensions at their first disagreement. Move the desired element leftwards in the first extension. Each intervening element is incomparable with it: either order relation would contradict one of the two extensions. Repeating this procedure makes the extensions equal. Every swap preserves the composed channel.

Records are included in gate supports. The theorem does not move a gate past another gate that reads its record.

Nor does it change a geometric parameter while treating the resulting channel as the same labelled gate.

Directional dependence

Proposition. In a square-lattice dependency model permitting at most one cardinal step of length a per tick, or a wait, the asymptotic velocity set is

$$\mathscr{V} = \{(v_x, v_y) : |v_x| + |v_y| \le a/\tau\}.$$

An isotropic physical speed a/τ is not dominated in every direction by this set.

Proof. After n ticks, $|\Delta x| + |\Delta y| \le na$. Conversely, suitable proportions of cardinal steps and waits approximate every velocity in the displayed diamond. But

$$\left(\frac{a}{\sqrt{2}\,\tau}, \frac{a}{\sqrt{2}\,\tau}\right)$$

has Euclidean norm a/τ and lies outside that diamond.

This proposition concerns the stated cardinal-hop model, not every block schedule. It illustrates why a directional inclusion theorem cannot be replaced by a comparison of two scalar speeds. More general periodic-graph velocity polytopes are studied in [Fritz2013].

Spectator-extension invariance

A spectator extension of $\rho_{\{\Gamma\}}$ to a disjoint interface N is any admissible state satisfying

$$\mathrm{Tr}_N \widetilde{\rho}_{\Gamma N} = \rho_\Gamma.$$

The extension may be entangled. Existing classical registers must remain classical in the full joint state.

Lemma. For a channel $\Phi: L(\mathscr{H}_{\{\Gamma\}}) \to L(\mathscr{H}_{\{\Gamma'\}})$,

$$\mathrm{Tr}_N((\Phi \otimes \mathrm{id}_N)(\widetilde{\rho})) = \Phi(\mathrm{Tr}_N \widetilde{\rho}).$$

Proof. For every $A \in L(\mathscr{H}_{\{\Gamma'\}})$,

$$\begin{aligned}\mathrm{Tr}\left(A\mathrm{Tr}_N[(\Phi \otimes \mathrm{id}_N)(\widetilde{\rho})]\right) &= \mathrm{Tr}\,((\Phi^*(A) \otimes I_N)\widetilde{\rho}) \\ &= \mathrm{Tr}\,(\Phi^*(A)\mathrm{Tr}_N \widetilde{\rho}) \\ &= \mathrm{Tr}\,(A\Phi(\mathrm{Tr}_N \widetilde{\rho})).\end{aligned}$$

Equality against every output observable proves the result.

Theorem. Consider a finite certified execution

$$C_k = (g_k; \Gamma_k; \rho_k; \mathrm{loc}_k; h_k), \qquad 0 \le k \le T,$$

and an admissible initial extension satisfying , with the same geometry, finite control state, and original-name locations. Assume:

• each original matter map in the extended execution is tensored with id_N;

• each coupling algebra is embedded as $\mathscr{A}_v \otimes I_N$, or its input is a specified function only of the original reduced state;

• spectator-blind control and resources, , hold;

• spectator locations have the required certified transport, and no spectator outcome is used during the comparison.

Then a paired extended execution is legal and, at every step,

$$\widetilde{g}_k = g_k, \qquad \widetilde{h}_k = h_k, \qquad \mathrm{Tr}_N \widetilde{\rho}_k = \rho_k.$$

Original-name locations also agree.

Proof. The assertions hold initially. Assume they hold at step k. By , both executions select the same rule, guard result, support, and resource certificate.

For a geometric rule, the geometric and resource inputs agree. Its output geometry and original-name transport

therefore agree. The matter states are unchanged.

For a matter rule, the geometric parameter is common. By ,

$$\mathrm{Tr}_N \widetilde{\rho}_{k+1} = \Phi_{g_k}(\mathrm{Tr}_N \widetilde{\rho}_k) = \Phi_{g_k}(\rho_k) = \rho_{k+1}.$$

For a coupling rule, every $A \in \mathscr{A}_v$ has equal expectation:

$$\mathrm{Tr}\,((A \otimes I_N)\widetilde{\rho}_k) = \mathrm{Tr}\,(A\rho_k).$$

The rule receives identical state functionals and returns identical certified geometry and original-name transport. The same conclusion holds when its input is a specified function of the original reduction.

The common control selection gives the same successor control state. Admissibility and the transport certificates establish legality of the next paired configurations. Induction completes the proof.

Corollary. Under the hypotheses of , inserting any finite product spectator state $\rho_0 \otimes \varphi_N$ leaves the geometric trace and original accessible matter reductions unchanged.

Proof. A product insertion is a special case of .

The capacity leak

Example. Let $g \in \{0, 1\}$, and let a geometric rule request $g \mapsto 1$. Suppose its capacity guard is

$$|\Gamma| \le K.$$

At $|\Gamma| = K$, the update is permitted. Adding a single product spectator blocks it, even when every coupling algebra is $\mathbb{C}I$ and no matter operation touches the spectator.

The example violates ; it does not contradict . Similar leaks can occur through queue priority, repartitioning, support selection, or a spectator-dependent choice of successor location.

Within the theorem's rule class, direct inspection of spectator state or cardinality is excluded as well. It would be inaccurate to infer that every possible leak is a capacity leak. The point is that coupling blindness alone is insufficient: all control and resource observations must also be declared.

Terminal readout and selective states

A spectator may be returned or consumed by terminal *Enter*. The partial trace in describes the accessible observation; it is not a term-level weakening rule.

Nonselective terminal readout of an entangled spectator preserves the original reduced state. Its conditional states need not be preserved. For

$$|\Omega\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

readout of the second factor leaves the first marginal $I/2$ when the outcome is ignored, but conditioning on the outcome gives $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$.

Thus the theorem neither identifies the enlarged joint output with the original output nor permits a later geometric rule to read the spectator outcome without changing the comparison.

Restricted destructive deferred readout

Immediate *Enter_S* removes S, whereas coherent recording retains it. Their outputs can be compared only after specifying a common accessible interface.

Lemma. For every state on $S \otimes B$,

$$\mathrm{Tr}_S\,(V\rho V^\dagger) = \sum_z |z\rangle\langle z|_A \otimes M_z \rho M_z^\dagger.$$

Proof. Write

$$\rho = \sum_{z,z'} |z\rangle\langle z'|_S \otimes \rho_{zz'}.$$

Then

$$V\rho V^\dagger = \sum_{z,z'} |z\rangle\langle z'|_S \otimes |z\rangle\langle z'|_A \otimes \rho_{zz'}.$$

Tracing out S removes precisely the terms with $z \neq z^{\{\prime\}}$.

Let $\mathcal{N}_A$ denote destructive computational-basis readout of the carrier A, returning a classical record R.

Lemma. For every channel $\Lambda: L(\mathcal{H}_B) \to L(\mathcal{H}_{B^{\{\prime\}}})$,

$$\mathrm{Tr}_S\left[\mathcal{N}_A((\mathrm{id}_{SA}\otimes\Lambda)(V\rho V^\dagger))\right] = (\mathrm{id}_R\otimes\Lambda)\mathcal{M}_S(\rho).$$

Proof. A channel acting only on B commutes with restriction over S and with carrier readout. Apply and identify the carrier's diagonal basis with the classical record basis.

Theorem. Start from geometry g_0 and a state ρ on $S \otimes B$. Compare:

• immediate $Enter_S$, followed by a finite continuation;

• coherent recording into a fresh A, the corresponding continuation, and terminal $Enter_A$. During the continuation assume:

• matter commands act only on the current complement B_k, never on the retained S, carrier A, or immediate record R;

• coupling inputs and state-dependent guards depend only on the reduced state on B_k;

• geometric rules inspect only their declared geometric labels;

• control, support, and resources are blind to the excluded registers, as in . Align executions by their common continuation events. Their geometric traces and complement reductions coincide at every aligned event. After terminal carrier readout, restricting the deferred output away from the retained factor S gives the immediate output on $R \otimes B_T$.

Proof. Immediately after the two initial operations,

$$\mathrm{Tr}_R\,\mathcal{M}_S(\rho) = \mathrm{Tr}_S\,\rho, \qquad \mathrm{Tr}_{SA}\,(V\rho V^\dagger) = \mathrm{Tr}_S\,\rho.$$

Thus the complement reductions agree.

Suppose they agree at a continuation event, together with geometry and control. Conditions (D2)–(D4) and give the same guard result, rule, resource certificate, and geometric update. Condition (D1) gives the same matter channel on the complement. By , its application preserves equality of complement reductions. Induction establishes the common trace.

For this fixed pair of inputs, compose the matter channels selected along the common trace into a channel Λ on the continuation interface. Equation gives equality of the final outputs on $R \otimes B_T$.

Remark. The terminal operation has already consumed the carrier A. The extra quantum output is the retained original factor S. It is S, not A, that is omitted from the common observation interface.

The selected Λ can depend on the input through the coupled geometric trace. The theorem is pairwise and does not assert that the complete coupled map is linear.

The deferred circuit must still return S or consume it in a well-typed continuation. Observational inaccessibility does not create a discard rule. Allowing intermediate access to R, S, or A requires a different theorem. Standard deferred-measurement constructions can treat more general record-dependent circuits [GurevichBlass2022], but their hypotheses are not silently imported into the present geometric comparison.

Record compatibility, residuals, and composition

This section concerns genuine finite-dimensional channels. It does not regard the nonlinear coupling in as a channel.

Effect pullback

Let

$$T : L(\mathcal{H}_{\mathrm{f}}) \to L(\mathcal{H}_{\mathrm{c}})$$

be a channel. It may include prescribed evolution followed by a coarse-graining channel. Let *{F_z: z ∈ Z}* be a fine POVM, *{E_w: w ∈ W}* a coarse POVM, and *q: Z → W* a label map. Define

$$\Pi_w = \sum_{z\,:\,q(z)=w} F_z.$$

Empty sums are zero. The two record laws are

$$p_T(w \mid \rho) = \mathrm{Tr}\,(E_w T(\rho)), \qquad p_{\mathrm{f}}(w \mid \rho) = \mathrm{Tr}\,(\Pi_w \rho).$$

Theorem. The two record laws agree for every state if and only if

$$T^*(E_w) = \Pi_w \qquad (w \in W).$$

Proof. Their difference at outcome *w* is

$$\mathrm{Tr}\,(\rho[T^*(E_w) - \Pi_w]).$$

The bracket is Hermitian. If its expectation vanishes in every state, testing its eigenvectors shows that every eigenvalue is zero. The reverse implication is the adjoint identity.

For example, if *T = Tr_r* and *Π_w = E_w ⊗ I_r*, compatibility is exact because *T^{*}(E) = E ⊗ I_r*. Readout at another time must first be pulled back through the appropriate evolution; a label aggregation is not an evolution map.

The exact unassisted record error

Set

$$D_w = T^*(E_w) - \Pi_w, \qquad D_A = \sum_{w \in A} D_w.$$

These operators are Hermitian and satisfy *∑_w D_w = 0*. Define

$$\delta_{\mathrm{rec}} = \sup_{\rho \in D(\mathscr{H}_{\mathrm{f}})} d_{\mathrm{TV}}(p_T(\cdot \mid \rho), p_{\mathrm{f}}(\cdot \mid \rho)).$$

Theorem. With the conventions in ,

$$\text{boxed}\ \delta_{\mathrm{rec}} = \max_{A \subseteq W} \|D_A\|_\infty \le \frac{1}{2}\sum_w \|D_w\|_\infty.$$

The maximum is attained by a pure input state.

Proof. For real numbers *d_w* with zero sum,

$$\frac{1}{2}\sum_w |d_w| = \max_{A \subseteq W} \Big|\sum_{w \in A} d_w\Big|.$$

Use *d_w = Tr (ρD_w)*. Since the event set is finite,

$$\delta_{\mathrm{rec}} = \max_{A \subseteq W} \sup_{\rho} |\mathrm{Tr}\,(\rho D_A)|.$$

For a Hermitian operator *H*, the latter supremum is *‖H‖_{∞}*, attained at an eigenvector whose eigenvalue has maximal absolute value. This proves the equality and attainment. The bound follows from *|Tr (ρD_w)| ≤ ‖D_w‖_{∞}*.

Example. On a one-dimensional input space, take the two four-outcome POVMs

$$p = (1/2, 1/2, 0, 0), \qquad q = (0, 0, 1/2, 1/2).$$

Their total-variation distance is *1*, whereas the largest individual-outcome discrepancy is *1/2*. The event *{1, 2}* attains the correct value. Thus *max_w ‖D_w‖_{∞}* is not the general formula. For two outcomes, it does suffice because *D_1 = −D_0*.

Example. Let *T(ρ) = HρH*, where *H* is the Hadamard unitary. Use computational-basis measurements before and after *T*, with identity label aggregation. Then

$$D_0 = \frac{X - Z}{2}, \qquad D_1 = -D_0, \qquad \delta_{\mathrm{rec}} = \frac{1}{\sqrt{2}}.$$

The state $|0\rangle\langle 0|$ gives error *1/2*, so it is not a worst-case input.

The optimization in is unassisted. Equation is not a diamond-norm formula.

Untouched entangled references

Let $\mathscr{M}_E$ and $\mathscr{M}_{\Pi}$ denote the record-only measurement channels associated with E_w and Π_w.

Corollary. Under , for every finite reference system N,

$$(\mathscr{M}_E \circ T) \otimes \mathrm{id}_N = \mathscr{M}_\Pi \otimes \mathrm{id}_N.$$

In particular, the joint record–reference outputs agree for entangled inputs.

Proof. For every operator X,

$$(\mathscr{M}_E \circ T)(X) = \sum_w |w\rangle\langle w| \operatorname{Tr}(T^*(E_w)X) = \sum_w |w\rangle\langle w| \operatorname{Tr}(\Pi_w X) = \mathscr{M}_\Pi(X).$$

Equal linear maps remain equal after tensoring with an identity.

An untouched reference is not the residual output produced by an instrument. Equal record channels determine the former comparison, not the latter. Also, an approximate unassisted bound does not become an ancillary-assisted bound merely because exact equality tensors.

Accessible output algebras

For a unital residual algebra $\mathscr{B} \subseteq L(\mathscr{H}_c)$, define

$$\mathscr{O}_{\mathscr{B}} = \left\{ \sum_w |w\rangle\langle w|_R \otimes B_w : B_w \in \mathscr{B} \right\}.$$

The record register is classical. Thus $\mathscr{O}_{\mathscr{B}}$ is the named record–residual observation algebra.

For channels Θ, Ξ with a common output space and a unital output algebra $\mathscr{O}$, define

$$d_{\mathscr{O}}(\Theta, \Xi) = \frac{1}{2} \sup_{\rho} \sup_{\text{substack } A = A^\dagger \in \mathscr{O}}$$

Proposition. The channels agree on $\mathscr{O}$ for every input if and only if

$$\Theta^*(A) = \Xi^*(A) \qquad (A \in \mathscr{O}).$$

Moreover,

$$d_{\mathscr{O}}(\Theta, \Xi) = \frac{1}{2} \sup_{\text{substack } A = A^\dagger \in \mathscr{O}}$$

Proof. For Hermitian A, equality of all input expectations is equality of the Hermitian adjoint images. General algebra elements are complex linear combinations of Hermitian elements. For the distance, interchange the two suprema and optimize the Hermitian expectation over states.

For $\mathscr{O}_{\mathbb{C}I}$, this is record-law comparison. For $\mathscr{O}_{L(\mathscr{H}_c)}$, it compares complete classical-record instrument outputs. Intermediate algebras specify intermediate observation boundaries.

Record agreement does not determine residuals

Now let the fine measurement be a PVM $\{P_z\}$. Define

$$\mathscr{F}(X) = \sum_w |w\rangle\langle w|_R \otimes T\Big(\sum_{z\,:\,q(z)=w} P_z X P_z \Big).$$

For a coarse instrument with outcome maps $\mathscr{I}_w$, put

$$\mathscr{G}(X) = \sum_w |w\rangle\langle w|_R \otimes \mathscr{I}_w(T(X)), \qquad E_w = \mathscr{I}_w^*(I).$$

These channels have the same record–residual output interface.

Theorem. Record compatibility does not imply agreement of $\mathscr{F}$ and $\mathscr{G}$ on $\mathscr{O}_{L(\mathscr{H}_c)}$, even when $T = id$.

Proof. Take one qubit, computational projectors P_0, P_1, and a single coarse outcome *. Let the coarse instrument be the identity. Then

$$E_* = I = \Pi_*,$$

so the record error is zero. For $\rho = |+\rangle\langle+|$,

$$\mathscr{G}(\rho) = |\langle|\rangle_R \otimes |+\rangle\langle+|, \qquad \mathscr{F}(\rho) = |\langle|\rangle_R \otimes I/2.$$

Their difference has residual block *X/2*, with eigenvalues *±1/2*. Therefore

$$\|\mathscr{G}(\rho) - \mathscr{F}(\rho)\|_1 = 1, \qquad d_{\mathrm{tr}}(\mathscr{G}(\rho), \mathscr{F}(\rho)) = \frac{1}{2}.$$

The full residual algebra distinguishes the outputs.

If two implementations approximate these targets on this input with trace-distance errors at most *ε_f* and *ε_c*, their mutual trace distance is at least

$$\frac{1}{2} - \varepsilon_{\mathrm{f}} - \varepsilon_{\mathrm{c}}.$$

This follows from the triangle inequality. More accurate implementation cannot eliminate an incompatibility of the ideal targets.

Sharp-sector intertwining

Assume now that *{E_w}* is also a PVM and the coarse instrument is the Lüders instrument

$$\mathscr{I}_w(X) = E_w X E_w.$$

The aggregated *Π_w = ∑_{q(z) = w} P_z* are mutually orthogonal projections.

Lemma. If *T^{*}(E_w) = Π_w*, then every Kraus operator *K_{α}* in any Kraus representation of *T* satisfies

$$E_w K_\alpha = K_\alpha \Pi_w.$$

Consequently,

$$E_w T(X) E_w = T(\Pi_w X \Pi_w) \qquad (X \in L(\mathscr{H}_{\mathrm{f}})).$$

Proof. For every vector *v*,

$$\sum_\alpha \|(I - E_w) K_\alpha \Pi_w v\|^2 = \langle v, \Pi_w T^*(I - E_w)\Pi_w v\rangle$$

$$= \langle v, \Pi_w (I - \Pi_w)\Pi_w v\rangle = 0.$$

Every summand is nonnegative. Hence *(I − E_w)K_{α}Π_w = 0* for every *α*. Similarly,

$$\sum_\alpha \|E_w K_\alpha (I - \Pi_w) v\|^2 = \langle v, (I - \Pi_w) T^*(E_w)(I - \Pi_w) v\rangle$$

$$= 0,$$

so *E_wK_{α}(I − Π_w) = 0*. These two identities give

$$E_w K_\alpha = E_w K_\alpha \Pi_w = K_\alpha \Pi_w.$$

Substitution into the Kraus representation proves .

Define the two dephasing channels

$$\mathscr{D}_P(X) = \sum_z P_z X P_z, \qquad \mathscr{D}_\Pi(X) = \sum_w \Pi_w X \Pi_w.$$

Theorem. Assume that the fine and coarse measurements are PVMs, the coarse instrument is Lüders, and *T^{*}(E_w) = Π_w* for every *w*. Then the following statements are equivalent:

• *$\mathscr{F}$ = $\mathscr{G}$*, equivalently they agree on *$\mathscr{O}$_{L(ℋ_c)}* for every input;

•

$$T \circ \mathscr{D}_\Pi = T \circ \mathscr{D}_P;$$

• for every *X* and every distinct *z*, *z^{'}* with *q(z) = q(z^{'})*,

$$T(P_z X P_{z'}) = 0.$$

Proof. By , the difference of outcome maps is

$$\mathcal{G}_w(X) - \mathcal{F}_w(X) = T\Big(\Pi_w X \Pi_w - \sum_{z\,:\,q(z)=w} P_z X P_z\Big)$$
$$= T\Big(\sum_{\text{substack } z \neq z'}$$

Each such difference is supported in the *E_w*-corner. Its sum over *w* is

$$T(\mathcal{D}_{\Pi}(X)) - T(\mathcal{D}_{P}(X)).$$

Since the *E_w* are orthogonal, this sum vanishes if and only if every corner vanishes: multiply the sum on both sides by *E_w*. This proves the equivalence of (i) and (ii).

Condition (iii) makes every summand in zero, proving (i). Conversely, assume (ii) and take *X = P_zYP_{z^{'}}*, with *z ≠ z^{'}* in the same coarse outcome. Then *𝒟_{Π}(X) = X* and *𝒟_P(X) = 0*, so *T(X) = 0*. This proves (iii).

Corollary. Under the hypotheses of :

• if *q* is injective, exact record compatibility already implies full Lüders compatibility;

• if there is one coarse outcome, full compatibility is equivalent to $T = T \circ \mathcal{D}_P$.

Proof. For injective *q*, the two dephasings in coincide. For one coarse outcome, *Π_{*} = I*, so *𝒟_{Π} = id*. Apply .

Thus $T = T \circ \mathcal{D}_P$ is sufficient in general but stronger than necessary. Coherence between distinct coarse outcomes need not be destroyed by *T*, because the coarse Lüders instrument removes it. The obstruction is precisely fine coherence within a single coarse outcome.

Compositional observation boundaries

The preceding comparisons concern specified output channels. For a continuation, a further question is whether input indistinguishability remains indistinguishability.

Theorem. Let $\mathcal{A} \subseteq L(\mathcal{H}_{in})$ and $\mathcal{B} \subseteq L(\mathcal{H}_{out})$ be unital ^*-subalgebras, and let Θ be a channel. On the unrestricted matrix-state spaces, the following are equivalent:

• for every pair of states,

$$\omega_\rho|_{\mathcal{A}} = \omega_\sigma|_{\mathcal{A}} \quad \Rightarrow \quad \omega_{\Theta(\rho)}|_{\mathcal{B}} = \omega_{\Theta(\sigma)}|_{\mathcal{B}};$$

•

$$\Theta^*(\mathcal{B}) \subseteq \mathcal{A}.$$

Proof. Condition (ii) implies (i) by pulling each output observable back.

Conversely, assume (i). Let *H = H^{†}* satisfy

$$\mathrm{Tr}\,(HA) = 0 \qquad (A \in \mathcal{A}).$$

Since $I \in \mathcal{A}$, *Tr H = 0*. With *d = dim ℋ_in*, choose *ε > 0* small enough that

$$\rho_{\pm} = I/d \pm \varepsilon H$$

are positive. They are states agreeing on $\mathcal{A}$. For every Hermitian $B \in \mathcal{B}$, assumption (i) gives

$$0 = \mathrm{Tr}\,(B[\Theta(\rho_+) - \Theta(\rho_-)]) = 2\varepsilon \mathrm{Tr}\,(H\Theta^*(B)).$$

Thus *Θ^{*}(B)* is orthogonal, in the real Hilbert space of Hermitian matrices, to the orthogonal complement of the Hermitian part of $\mathcal{A}$. Finite-dimensional double orthogonal complementation puts it in that Hermitian part. Complex linearity proves $\Theta^{*}(\mathcal{B}) \subseteq \mathcal{A}$.

If successive continuations obey *Θ_k^{*}(𝒜_{k + 1}) ⊆ 𝒜_k*, their composition preserves the initial observation boundary. This follows by composing the adjoint inclusions.

Classical block-diagonal state spaces

Let

$$\mathcal{C} = \mathrm{bigoplus}_j L(\mathcal{H}_j) \subseteq L\big(\mathrm{bigoplus}_j \mathcal{H}_j\big)$$

be a prescribed block-diagonal state algebra, and let

$$\mathcal{E}_{\mathcal{C}}(X) = \sum_j Q_j X Q_j$$

be its trace-orthogonal pinching projection. Admissible input states are the density operators in $\mathcal{C}$. The accessible input algebra $\mathcal{A}$ need not lie inside $\mathcal{C}$.

Proposition. For all admissible input pairs, agreement on $\mathcal{A}$ is preserved as agreement on $\mathcal{B}$ if and only if

$$\mathcal{E}_{\mathcal{C}}\Theta^*(\mathcal{B}) \subseteq \mathcal{E}_{\mathcal{C}}(\mathcal{A}).$$

Equivalently,

$$\Theta^*(\mathcal{B}) \subseteq \mathcal{A} + \mathcal{C}^{\perp},$$

where $\mathcal{C}^{\perp}$ is the Hilbert–Schmidt orthogonal complement.

Proof. An admissible state sees A only through $\mathcal{E}_{\mathcal{C}}(A)$. Therefore the displayed inclusion is sufficient.

For necessity, repeat the perturbation proof of with Hermitian $H \in \mathcal{C}$ annihilating $\mathcal{E}_{\mathcal{C}}(\mathcal{A})$. The states $I/d \pm \varepsilon H$ are admissible. The resulting orthogonality puts $\mathcal{E}_{\mathcal{C}}\Theta^*(B)$ in $\mathcal{E}_{\mathcal{C}}(\mathcal{A})$ for every Hermitian B, and then for every B. Finally, $\ker \mathcal{E}_{\mathcal{C}} = \mathcal{C}^{\perp}$, proving the equivalent formulation.

The image $\mathcal{E}_{\mathcal{C}}(\mathcal{A})$ need not itself be an algebra. The statement uses it as the actual observable linear space on the restricted states.

A dictionary for distributional observations

This section gives three linear inference rules for a separate distributional problem. It assumes no particular graph kernel and proves no finite-part existence theorem for a graph.

Let X, Y be smooth manifolds, with a fixed test-function convention. Write $\mathcal{D}(X) = C_c^{\infty}(X)$ and $\mathcal{D}'(X)$ for complex distributions, using the complex-bilinear distribution–test pairing. Convergence below means distributional convergence.

The names VII-C and VII-D refer only to the separate Wick/graph track. Nothing from that track is needed to state or prove the results here.

Descent through declared invisible directions

In finite dimensions, the inclusion $\Theta^*(\mathcal{B}) \subseteq \mathcal{A}$ is dual to

$$\Theta(\mathcal{A}^{\perp}) \subseteq \mathcal{B}^{\perp}.$$

Indeed, both say that $\mathrm{Tr}(B\Theta(H)) = \mathrm{Tr}(\Theta^*(B)H) = 0$ whenever H annihilates $\mathcal{A}$. The following quotient criterion has the same formal role.

Proposition. Let $P: \mathcal{D}'(X) \to \mathcal{D}'(Y)$ be continuous and linear, and let $\mathfrak{C} \subseteq \mathcal{D}'(X)$ and $\mathfrak{C}_P \subseteq \mathcal{D}'(Y)$ be declared real linear subspaces. The assignment

$$[u] \longmapsto [Pu]$$

is well defined on the real vector-space quotients if and only if

$$P\mathfrak{C} \subseteq \mathfrak{C}_P.$$

With quotient topologies, the induced map is continuous.

Proof. If $u - v \in \mathfrak{C}$, then $Pu - Pv = P(u - v)$. Condition therefore makes the output class independent of the representative. Conversely, well-definedness applied to $[c] = [0]$, $c \in \mathfrak{C}$, gives $Pc \in \mathfrak{C}_P$. Continuity follows from the defining property of the quotient topology.

For $\mathfrak{C} = \mathbb{R}r$, this is a real quotient, even though the ambient distributions are complex. If source and target use the same nonzero residue r, descent requires $Pr \in \mathbb{R}r$. Failure of this inclusion is failure of descent for that specified quotient; it is not a proof that the original finite part fails to exist.

No multiplication is defined on these quotients.

Counterterm-blind tests

Proposition. Let $r \neq 0$ and L belong to $\mathcal{D}'(X)$. Choose $\Psi \in \mathcal{D}(X)$ with $\mathrm{pair}\, r\Psi = 1$, and set

$$\Phi^{\circ} = \Phi - \operatorname{pair} r\Phi\, \Psi.$$

Then

$$L \in \mathbb{R}r \quad \Leftrightarrow \quad \begin{cases} \operatorname{pair} L\Phi^{\circ} = 0 & \text{for every} \Phi \in \mathscr{D}(X), \\ \operatorname{pair} L\Psi \in \mathbb{R}. \end{cases}$$

Proof. Nonzero r permits a test with nonzero pairing, which can be rescaled to obtain Ψ. By construction, *pair* $r\Phi^{\circ} = 0$.

If $L = cr$, $c \in \mathbb{R}$, both conditions follow. Conversely, if they hold, then

$$\operatorname{pair} L\Phi = \operatorname{pair} L\Phi^{\circ} + \operatorname{pair} r\Phi \operatorname{pair} L\Psi = c \operatorname{pair} r\Phi, \qquad c = \operatorname{pair} L\Psi \in \mathbb{R}.$$

This holds for every compact test, so $L = cr$.

For real distributions and real tests, the reality condition is automatic. Over complex tests it must not be omitted: tests annihilating r alone cannot distinguish real from nonreal multiples of r.

Suppose a complete regulator difference has already been shown to converge,

$$u_{\varepsilon}^{a} - u_{\varepsilon}^{b} \to L_{ab} \quad \text{in} \mathscr{D}'(X).$$

Then a single compact test Φ° with

$$\operatorname{pair} r\Phi^{\circ} = 0, \qquad \langle L_{ab}, \Phi^{\circ} \rangle \neq 0$$

obstructs removal of *L_ab* by an allowed finite counterterm in $\mathbb{R}r$. These tests are diagnostics. Existence of a finite part still means convergence against all compact tests, not only against *ker r*.

For an incidence representation, suppose a pushforward is defined and

$$r = \psi_{*}\nu, \qquad L_{ab} = \psi_{*}(\gamma_{ab}\nu).$$

For example, *γ_ab* may be smooth and ψ proper on the relevant supports. The correct condition is

$$L_{ab} = cr \quad \Leftrightarrow \quad \psi_{*}((\gamma_{ab} - c)\nu) = 0.$$

Nonconstancy of *γ_ab* by itself is not an obstruction unless the needed injectivity or separation property of the pushforward has also been proved.

Presentation defects before regulator removal

Fix $\mu > 0$, $\varepsilon > 0$, and

$$\ell_{\mu}(\varepsilon) = \log\left(\frac{1}{\mu\varepsilon}\right).$$

Let $P: \mathscr{D}'(X) \to \mathscr{D}'(Y)$ be continuous for distributional convergence.

Proposition. Suppose the two complete finite parts exist:

$$u_{\mathrm{fin}} = \lim_{\varepsilon \text{downarrow } 0} [u_{\varepsilon} - \ell_{\mu}(\varepsilon) r],$$
$$u_{\mathrm{fin}}^{(P)} = \lim_{\varepsilon \text{downarrow } 0} [u_{\varepsilon}^{(P)} - \ell_{\mu}(\varepsilon) r^{(P)}].$$

Then the presentation defect is

$$\mathscr{A}_P := u_{\mathrm{fin}}^{(P)} - P u_{\mathrm{fin}}$$
$$= \lim_{\varepsilon \text{downarrow } 0} \left[u_{\varepsilon}^{(P)} - P u_{\varepsilon} - \ell_{\mu}(\varepsilon)(r^{(P)} - Pr) \right].$$

Proof. Continuity of P gives

$$P[u_{\varepsilon} - \ell_{\mu}(\varepsilon) r] \to P u_{\mathrm{fin}}.$$

Subtract this convergence from and collect terms.

The identity separates:

• the finite-cutoff comparison $u_{\varepsilon}^{(P)} - P u_{\varepsilon}$;

• the residue comparison $r^{(P)} - Pr$;

• the finite normalization left after both subtractions. If both finite-cutoff and residue identities hold literally, then $\mathscr{A}_P = 0$. Otherwise their failures must be computed in the specified presentation. A formal unregulated identity is not a substitute.

Transported differentiation is an immediate application: a fixed differential operator is continuous on distributions, so it transports an established finite part together with its residue. Cubic parenthesization is different. Intermediate multiplication of singular distributions is not automatically a defined continuous linear map P.

Tests on a product manifold must not be restricted to a selected family $f \otimes g \otimes f^{\{\prime\}}$ without a proved extension argument. Convergence on simple product tests alone is not the asserted distributional convergence on all compact joint tests. The correct organization of a future graph estimate may separate regular regions, transition regions, matching, and uniform control near an incidence intersection such as $\mathscr{R} \cap \Delta_{13}$. That organization does not supply those estimates.

One certified four-face slicing exchange

We now give one complete geometric-rule certificate. The ambient spacetime is fixed; only its prescribed slicing and the associated cell incidence vary. No matter channel, readout, or coupling update fires at the event.

Fixed cone and oriented holonomy

Use Minkowski coordinates $z = (T, x, y)$ and

$$\eta = \operatorname{diag}(-1, 1, 1).$$

For a prescribed deficit $0 < \mu_c < 2\pi$, set $\alpha = 1 - \mu_c/(2\pi)$. The punctured cone metric is

$$ds^2 = -dT^2 + dr^2 + \alpha^2 r^2 d\theta^2, \qquad r > 0, \quad \theta \bmod 2\pi.$$

Its completion has a timelike conical axis. Its spinless point-particle interpretation is standard [DeserJackiwHooft1984]; here the metric is prescribed rather than derived.

On the universal cover, let

$$D(T, r, \tilde{\theta}) = (T, r\cos(\alpha\tilde{\theta}), r\sin(\alpha\tilde{\theta})), \qquad \mathrm{T}(\tilde{\theta}) = \tilde{\theta} + 2\pi.$$

With

$$R(\mu_c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\mu_c & -\sin\mu_c \\ 0 & \sin\mu_c & \cos\mu_c \end{pmatrix},$$

we have

$$D \circ \mathrm{T}^{-1} = R(\mu_c) D.$$

Choose angular chart lifts on

$$U_1 = (-\varepsilon, \pi + \varepsilon), \qquad U_2 = (\pi - \varepsilon, 2\pi + \varepsilon), \qquad 0 < \varepsilon < \pi/4.$$

Refine if needed to injective developing charts. The two overlap components have different transitions:

$$\Lambda^{\pi}_{2 \leftarrow 1} = I, \qquad \Lambda^{0}_{1 \leftarrow 2} = R(\mu_c).$$

The positively oriented particle loop has ordered affine word

$$J = (R(\mu_c), 0)(I, 0) = (R(\mu_c), 0).$$

The overlap-component index is essential. No flat chart is asserted to contain the axis.

These matrices act on spacetime tangent vectors. More generally, if

$$D \circ \mathrm{T}_\gamma^{-1} = h_\gamma D, \qquad h_\gamma(z) = L_\gamma z + b_\gamma,$$

a lifted parallel vector has constant developed components. At the endpoint, differentiation of this identity gives

$$dD_x^{\vee} d\mathrm{T}_\gamma^{-1}(v') = L_\gamma dD_{\mathrm{T}_\gamma x}^{\vee}(v') = L_\gamma dD_x^{\vee}(v).$$

Thus the tangent holonomy in the starting developed frame is *L_{γ}*. This is gluing data for the spacetime, not an unrelated internal fibre; compare the polygon-model formulations in [tHooft1992,Kadar2005].

For the certificate fix

$$\mu_{\mathrm{c}} = \pi/2.$$

Its value is independent of the slicing slopes introduced next. The future unit holonomy axis is *e_0 = (1, 0, 0)*. For a marked spacelike face with future unit normal *n*, define

$$\chi(p, n) = \operatorname{arccosh}(-\eta(p, n)).$$

Proper future-preserving chart changes transform both vectors and conjugate the holonomy, preserving this quantity. It is a particle–slice observable, not a particle invariant alone.

Four moving affine faces

Choose

$$p = \frac{3}{5}, \quad q = \frac{5}{13}, \quad a = \sqrt{1 - p^2} = \frac{4}{5}, \quad b = \sqrt{1 - q^2} = \frac{12}{13}, \quad \delta = b - a = \frac{8}{65}.$$

For $-1 \le s \le 1$, let

$$f_E = px + as, \qquad f_W = -px + as,$$
$$f_N = qy + bs, \qquad f_S = -qy + bs,$$
$$F_s(x, y) = \max\{f_E, f_W, f_N, f_S\}.$$

A core face is the region where the corresponding affine function attains the maximum, with its interior given by strict dominance.

Proposition. The graph *T = F_s(x, y)* is continuous, piecewise affine, and *p*-Lipschitz. Every face is spacelike, and for *s_2 > s_1*,

$$F_{s_2}(x, y) - F_{s_1}(x, y) \ge a(s_2 - s_1) > 0.$$

For *s < 0*, the central adjacency is *EW*, with edge

$$x = 0, \quad |y| \le \frac{\delta(-s)}{q}, \quad T = as, \qquad \ell_{-}(s) = \frac{2\delta}{q}(-s) = \frac{16}{25}(-s).$$

For *s > 0*, the central adjacency is *NS*, with edge

$$y = 0, \quad |x| \le \frac{\delta s}{p}, \quad T = bs, \qquad \ell_{+}(s) = \frac{2\delta}{p}s = \frac{16}{39}s.$$

The central incidence changes as

$$\{EWN, EWS\} \to \{EWNS\} \to \{NSE, NSW\}.$$

Proof. Each affine function has spatial gradient of norm at most *p < 1*. Their maximum has the same Lipschitz bound. On a face of slope *v*, a nonzero tangent vector has squared norm

$$\eta((v \cdot u, u), (v \cdot u, u)) = |u|^2 - (v \cdot u)^2 > 0.$$

Every affine function increases in *s* at rate at least *a*, proving the ordering bound.

For *EW*, equality of the two affine functions forces *x = 0*. Their common value *as* dominates the other two precisely when $q|y| \le -\delta s$. This gives and its triple endpoints. The endpoint difference has constant *T, x,* so its spacelike length is the displayed *y*-distance.

For *NS*, equality forces *y = 0*, and dominance is $p|x| \le \delta s$, giving .

At *s = 0, F_0 = max {p|x|, q|y|}*. All four faces meet only at the origin, and each contains an open strict-dominance cone. Thus the central edge collapses, but the two-dimensional patch does not.

The exchange is transverse:

$$\dot{\ell}_{-}(0^{-}) = -\frac{16}{25} < 0, \qquad \dot{\ell}_{+}(0^{+}) = \frac{16}{39} > 0.$$

The mixed adjacencies *EN*, *ES*, *WN*, *WS* persist. In their respective quadrants they satisfy

$$p|x| - q|y| = \delta s.$$

The only remaining core pairs are *EW* and *NS*, already determined. The exchange is forced by this prescribed maximum rule, not by a uniqueness theorem for all slicings of the spacetime.

Adapted charts and exact products

For each face, use the following Lorentz transformation:

$$C_E = \begin{pmatrix} 5/4 & -3/4 & 0 \\ -3/4 & 5/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_W = \begin{pmatrix} 5/4 & 3/4 & 0 \\ 3/4 & 5/4 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C_N = \begin{pmatrix} 13/12 & 0 & -5/12 \\ 0 & 1 & 0 \\ -5/12 & 0 & 13/12 \end{pmatrix}, \quad C_S = \begin{pmatrix} 13/12 & 0 & 5/12 \\ 0 & 1 & 0 \\ 5/12 & 0 & 13/12 \end{pmatrix}.$$

Direct multiplication gives *C_i^TηC_i = η*, determinant *1*, and future time orientation. For the slope *v_i* of face *i*,

$$(C_i z)^0 = \frac{T - v_i \cdot (x,y)}{\sqrt{1 - |v_i|^2}} = s$$

on that face.

Define *Λ_{i ← j} = C_iC_j^{−1}*. The opposite pairs are

$$\Lambda_{E \leftarrow W} = \begin{pmatrix} 17/8 & -15/8 & 0 \\ -15/8 & 17/8 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda_{N \leftarrow S} = \begin{pmatrix} 97/72 & 0 & -65/72 \\ 0 & 1 & 0 \\ -65/72 & 0 & 97/72 \end{pmatrix}.$$

Writing *X_{+} = E*, *X_{−} = W*, *Y_{+} = N*, *Y_{−} = S*, the mixed pairs are

$$\Lambda_{X_\varepsilon \leftarrow Y_v} = \begin{pmatrix} 65/48 & -\varepsilon\, 3/4 & v\, 25/48 \\ -\varepsilon\, 13/16 & 5/4 & -\varepsilon v\, 5/16 \\ v\, 5/12 & 0 & 13/12 \end{pmatrix}.$$

Reverse transitions are inverses. Core translations are zero.

Proposition. Every closed core transition word is the identity. In particular, the incoming words

$$\Lambda_{E \leftarrow W}\Lambda_{W \leftarrow N}\Lambda_{N \leftarrow E}, \qquad \Lambda_{E \leftarrow W}\Lambda_{W \leftarrow S}\Lambda_{S \leftarrow E}$$

and the outgoing words

$$\Lambda_{N \leftarrow S}\Lambda_{S \leftarrow E}\Lambda_{E \leftarrow N}, \qquad \Lambda_{N \leftarrow S}\Lambda_{S \leftarrow W}\Lambda_{W \leftarrow N}$$

equal *I*. The four-face event word also equals *I*.

Proof. Every word is a product

$$(C_{i_0}C_{i_1}^{-1})(C_{i_1}C_{i_2}^{-1})\cdots(C_{i_{m-1}}C_{i_0}^{-1}),$$

which telescopes.

The nonidentity adapted-frame transitions are pure changes of frame in one flat development. They do not create spacetime curvature.

A compact collar

Set *H = 2*, and cap the graph by

$$g_s(x,y) = \min\{F_s(x,y), H + s\}.$$

Its non-resting region is

$$\mathcal{R}_s = [-X_s, X_s] \times [-Y_s, Y_s], \qquad X_s = \frac{10+s}{3}, \quad Y_s = \frac{26+s}{5}.$$

Indeed, *F_s ≤ H + s* is equivalent to the two displayed coordinate bounds. Fix the outer rectangle

$$K = [-R_x, R_x] \times [-R_y, R_y], \qquad R_x = \frac{14}{3}, \quad R_y = 6.$$

Its collar margins are

$$R_x - X_s = \frac{4 - s}{3} > 0, \qquad R_y - Y_s = \frac{4 - s}{5} > 0.$$

Proposition. The capped graph is continuous, piecewise flat, *p*-Lipschitz, and strictly time-ordered with lower increment *a(s_2 − s_1)*. It equals $T = H + s$ on a neighbourhood of ∂K. The central edge laws in are unchanged.

Proof. Minimum and maximum preserve a common Lipschitz bound. Both arguments of the minimum increase by at least *a(s_2 − s_1)*, so the minimum does also. The strict margins in give the resting boundary neighbourhood.

On the incoming central edge, the cap-height margin is *2 + s/5 ≥ 9/5*; on the outgoing edge it is *2 + s/13 ≥ 25/13*. Hence neither central edge is clipped.

There are four core faces inside $\mathcal{R}_s$. Partition $K \setminus \mathcal{R}_s$ into four resting collar polygons. The north-east polygon has counterclockwise spatial vertices

$$(X_s, 0),\ (R_x, 0),\ (R_x, R_y), ?(0, R_y),\ (0, Y_s),\ (X_s, Y_s).$$

The other three are reflections, ordered counterclockwise. Every collar vertex has $T = H + s$. The inner and outer side midpoints divide each rectangular boundary into eight segments. These coplanar subdivisions are part of the declared cell interface.

For affine pairs, use

$$(L, b)z = Lz + b, \qquad (L, b)(L', b') = (LL', Lb' + b).$$

The collar chart is

$$A_0(z) = z - He_0 = (T - H, x, y).$$

The core chart is *A_i = (C_i, 0)*, so a core-to-collar transition is

$$A_0 A_i^{-1} = (C_i^{-1}, -He_0).$$

Collar-to-collar transitions are identities.

Proposition. Every collar or mixed core–collar vertex word is the affine identity. All cells close and all matching edges have identical induced lengths.

Proof. Every transition is *A_iA_j^{−1}* for globally defined affine isometries of the patch, so closed words telescope in the affine group.

Core cells are intersections of their affine dominance half-spaces with $\mathcal{R}_s$. The collar polygons are the explicitly listed simple polygons and their reflections. Adjacent cells share the same developed endpoints and segment. Affine isometries preserve its induced length.

Complete vertices, lengths, and isolation

The central endpoints are those of . All other vertices have $T = H + s$ and spatial coordinates in the following list:

$$(\varepsilon X_s, \nu Y_s), \quad (\varepsilon X_s, 0), \quad (0, \nu Y_s), \quad (\varepsilon R_x, \nu R_y), \quad (\varepsilon R_x, 0), \quad (0, \nu R_y), \qquad \varepsilon, \nu \in \{-1, 1\}.$$

The four mixed core edges join their appropriate central triple junctions to the corresponding inner corners. Their common length is

$$L_{\text{arm}}(s) = \begin{cases} \frac{10 + s}{75}\sqrt{1921}, & -1 \le s \le 0, \\ \frac{26 + s}{195}\sqrt{1921}, & 0 \le s \le 1. \end{cases}$$

The two expressions agree at *s = 0*.

Proposition. The list in is complete. Every actual noncentral edge has length greater than *1/2* throughout *[−1, 1]*. All eight faces remain two-dimensional, and the only incidence event in the patch is the central exchange.

Proof. For the incoming north-east arm, the displacement from the central endpoint to the inner corner is

$$X_s(p, 1, p/q).$$

Its squared Minkowski norm is

$$X_s^2(1 + (p/q)^2 - p^2) = X_s^2 \frac{1921}{625}.$$

For the outgoing arm the displacement is *Y_s(q, q/p, 1)*, with norm *Y_s√(1921)/39*. These give ; symmetry gives the other arms.

The remaining edges lie in the resting plane and have the Euclidean lengths listed in the table. The smallest such length over the window is *3/5 > 1/2*. The arms have length at least *3√(1921)/25 > 1/2*.

The central endpoints remain strictly inside $\mathcal{R}_s$. Each core face contains a strict-dominance neighbourhood just inside its inner side midpoint. Each collar polygon contains a nonempty open region because the margins are positive. The mixed arms and all peripheral incidences therefore persist. The central laws determine the only exchange.

No general assertion that affine endpoint motion always gives affine length is used. Here the arm directions are fixed on each epoch, and all other lengths follow directly from their displayed segments.

Attachment to one frozen conical exterior

Use the cone wedge of angle *3π/2*, and place the patch in its first quadrant by

$$(X, Y) = (20 + x, 20 + y).$$

The translated K lies strictly inside that quadrant, away from the axis and seam. On the rest of the cone use $T = H + s$. By , the pieces agree on an open boundary neighbourhood.

Here is a finite exterior cellulation with exactly the required patch boundary subdivision. Put

$$A = \{20 - R_x, 20, 20 + R_x\}, \qquad B = \{20 - R_y, 20, 20 + R_y\}.$$

In the first quadrant cut by $X \in A$ and $Y \in B$. In the second quadrant, using coordinates *(−X, Y)*, use cuts *(−X)* $\in A$, $Y \in B$. In the third quadrant, using *(−X, −Y)*, use cuts *(−X)* $\in A$, *(−Y)* $\in A$. The positive Y-axis subdivisions match through B, the negative X-axis subdivisions match through A, and the two wedge-seam subdivisions both use A. Keep the cells outside the translated K and replace its four grid rectangles by the eight patch cells. This gives finitely many cell labels; unbounded exterior cells are allowed.

Use shifted cone developments

$$D_H = (T - H, r\cos(\alpha\tilde{\theta}), r\sin(\alpha\tilde{\theta})).$$

Time translation commutes with the seam rotation, so the particle word is unchanged. From a collar chart to the first-quadrant cone development, the transition is the fixed spatial translation

$$(I, (0, 20, 20)).$$

Proposition. Both event germs use the same boundary labels, eight fixed spatial boundary segments, constant boundary lengths, resting boundary time law $T - H = s$, and fixed outgoing transitions. The exterior cellulation, cone metric, and resting particle neighbourhood are not rewritten.

On both sides of the event,

$$\mu_c = \pi/2, \qquad p_{\text{particle}} = e_0, \qquad \chi(e_0, e_0) = 0, \qquad J = (R(\pi/2), 0).$$

Proof. The construction matches the same resting plane near the same fixed boundary. The exterior grid is independent of s in spatial coordinates, and its affine time law is prescribed once.

A particle loop can be chosen in the unchanged neighbourhood of the axis and has exactly the word . A path that enters the patch acquires a telescoping affine word determined by its endpoint charts. Changing internal incidence cannot change that word. The particle's slice normal remains *e_0*, proving the remaining claims.

``Frozen'' here means that the declared exterior labels and affine-time laws are not changed by the event. It does not mean that the time coordinate fails to advance with s.

Transition stratum, exact guard, and transport

Let $\mathscr{Q}_{-}$ and $\mathscr{Q}_{+}$ be the regular families for $-1 \le s < 0$ and $0 < s \le 1$. Let $\mathscr{Q}_0^{-}$ and $\mathscr{Q}_0^{+}$ be their certified event germs.

Both germs realize the same capped surface at $s = 0$, with one four-face junction. The incoming germ retains an *EW* edge slot c_{-} and its coincident endpoint labels j_N, j_S. The outgoing germ retains an *NS* slot c_{+} and its coincident endpoint labels j_E, j_W. These are event metadata, not positive length edges.

Include a mode bit and a nonnegative permission label. The semantic event rule

$$G_{\mathrm{ex}} : \mathscr{Q}_0^{-} \to \mathscr{Q}_0^{+}$$

has guard

$$\text{mode} = 0, \qquad \text{perm} \ge 1, \qquad s = 0.$$

It changes mode to 1, decreases permission by 1, and selects outgoing incidence. It does not advance s. The no-other-collapse certificate is , not a numerical test.

No active physical matter is placed inside K in this example. Inaccessible spectator names may nevertheless have abstract cell labels there. Their transport is the fixed map

$$\tau(c_{-}) = \tau(j_N) = \tau(j_S) = c_{+},$$

with persistent face, edge, vertex, collar, boundary, and exterior labels unchanged. This map factors through the event junction. It does not inspect states or infer developed positions.

Proposition. The event is a certified G-rule. Its geometric guard, support, permission update, and original-name transport are spectator-blind under separately reserved spectator location and history storage. Thus holds at the semantic event boundaries.

Proof. The one-sided families, common event realization, all local products, positive peripheral lengths, and fixed boundary data have been proved in . They certify the outgoing germ and its continuation.

The guard uses only the exact event parameter, mode, permission, and the fixed family certificate. The transport is a fixed function of cell labels. None of these operations tests matter state or name cardinality. Separate reservation makes additional spectator storage irrelevant to the geometric capacity check and leaves original-name transport unchanged.

Finite code and reversible implementation

Exact timing versus numerical reports. If an incoming length is $v(-s)$, floor quantization at step h reports zero throughout $0 \le -s < h/v$. A zero-report guard can therefore fire prematurely.

Fix an integer $d \ge 0$ and store the exact parameter

$$s = j2^{-d}, \qquad -2^{d} \le j \le 2^{d},$$

in a signed $(d + 2)$-bit word. The event tests $j = 0$.

For report precision $b_r \ge 1$, store each length in an unsigned $(b_r + 4)$-bit word:

$$\widehat{L} = 2^{-b_{\mathrm{r}}} \lfloor 2^{b_{\mathrm{r}}} L \rfloor, \qquad 0 \le L - \widehat{L} < 2^{-b_{\mathrm{r}}}.$$

Every length is less than 16, so four integer bits suffice. Both central slots are retained; the inactive one is reported as zero. At the event both are zero.

The exact decoder reconstructs geometry from the fixed family, the mode, and exact s, not from independent numerical edge labels. Reports must agree with on its domain. The irrational arm reports can be precomputed exactly on the finite grid by comparisons of squared rational numbers.

Fixed chart data. The code includes identifiers for the charts in , the collar chart, the prescribed transitions, the fixed exterior translation, and $R(\pi/2)$. All entries are rational. Only these listed data and products are needed; no finite closure under arbitrary Lorentz multiplication is asserted.

Location and history words. Fix a finite named matter interface of size n. Let ℓ bits accommodate the union of pre- and post-event cell labels. Encode each location in an ℓ-bit word and provide a second ℓ-bit history word,

initially zero. Invalid binary labels lie outside the decoder domain.

Let x be the tuple of valid incoming location words. On the forward event domain, prescribe

$$(0, k, j = 0, \text{reports}, x, 0_{\text{hist}})$$
$$\text{longmapsto}\,(1, k-1, j = 0, \text{reports}, \tau(x), x_{\text{hist}}), \qquad 1 \le k \le 2^r - 1,$$

where the permission word has $r \ge 1$ bits. All fixed family identifiers are retained.

Lemma. The map is injective, and its input and output sets are disjoint.

Proof. The output history determines x, and the output permission determines k. All other words are retained. Input and output modes differ.

Proposition. Exchange each valid input in with its specified output, and fix every other basis string. This defines a permutation *U_ex* of the complete finite code space. On the declared forward domain,

$$\text{Dec}\,(U_{\text{ex}}c) = G_{\text{ex}}(\text{Dec}\,(c)).$$

The decoded equality includes geometry, permission, event germ, and original-name locations. History remains an output of the implementation.

Proof. By , all exchanged pairs are disjoint. Their exchanges and the identity on the remaining strings form a permutation.

At $j = 0$, both central lengths are zero; the two arm formulas agree; and every peripheral formula is identical on the two germs. Hence the unchanged report words remain valid. The mode selects outgoing incidence, permission decreases, all fixed chart data remain unchanged, and the location tuple becomes $\tau(x)$. These are precisely the semantic event outputs.

Ignoring history in the geometric decoder does not discard it from the circuit interface.

Proposition. The permutation *U_ex* has an exact finite NOT–CNOT–Toffoli realization with clean work bits returned to zero. Code, location, and history wires are returned. The matter density operator is acted on by the identity.

Proof. Every finite permutation is a product of transpositions. For two M-bit strings, choose a Gray path $v_0, \ldots, v_D$ between them, changing one differing bit at a time. If $s_j = (v_j\ v_{j+1})$, the product

$$s_0 s_1 \cdots s_{D-2} s_{D-1} s_{D-2} \cdots s_1 s_0$$

exchanges the endpoints and fixes the intermediate strings.

An adjacent transposition is a target NOT controlled on a specified pattern of the other bits. Negative controls use surrounding NOT gates. A multiple-control NOT is implemented by computing the conjunction of its controls into clean work bits with Toffoli gates, flipping the target, and reversing the conjunction computation. Zero, one, or two controls use NOT, CNOT, or Toffoli directly. At most M clean work bits suffice.

Apply this construction to a transposition decomposition of *U_ex*. Each gate consumes and returns its interface. The work is uncomputed, but the history required by is deliberately retained. Tensor the circuit with the identity on matter.

Excluding fixed table storage, a sufficient number of data bits is

$$M = 26(b_{\text{r}} + 4) + 1 + r + (d + 2) + 2n\ell.$$

The generic synthesis uses at most $2M$ bits including clean work. This is an existence bound, not an efficiency claim.

Valid location words and initially zero history are representation preconditions, not geometric guards that inspect spectator state. Additional spectators receive separately reserved location/history storage. The semantic theorem compares decoded event endpoints; it does not assert identical low-level gate lists for different interface sizes. Intermediate compiler states need not lie in the geometric decoder domain and are not exposed to the geometric controller. No claim about retiming an event through such intermediate states is made.

The reverse action of *U_ex* is a reversible completion, not an additional forward geometric rule. The forward rule still requires . The circuit implements the discrete event, not continuous affine evolution or arbitrary real

geometric data.

Certificate theorem and intrinsic slice geometry

Theorem. In the fixed conical spacetime of deficit *π/2*, the capped slicing has exactly one central adjacency exchange in *[−1, 1]*. Its certificate contains:

• spacelike, time-ordered slices and transverse incoming/outgoing edge laws;

• explicit adapted affine charts and exact ordered local products;

• an admitted event stratum and an exact parameter guard;

• a compact collar matching one fixed exterior cellulation;

• fixed abstract location transport on declared cell labels;

• spectator-blind control under separately reserved spectator location/history storage;

• a decoder-correct finite event permutation and exact reversible circuit with clean work uncomputed and matter unchanged. The particle data are those in throughout.

Proof. Items 1–4 follow from . The germ and transport construction and give items 3, 5, and 6. Item 7 is . The particle statement is .

The exchange has the four-region incidence of the type-A exchange in [tHooft1993]. Its scope is: Type-A incidence only. Not the transition catalogue. Not a particle hop. Not a field equation. Collar subdivisions are declared interface, not a minimal polygon gauge.

Although the patch has trivial ambient holonomy, its marked slice is not intrinsically flat at every vertex. The calculation in gives the event defect

$$\kappa_0 = 2\pi - 2\arccos(-1121/1921) - 2\arccos(671/1921) < 0.$$

Each incoming triple junction and each outgoing triple junction has defect *κ_0/2*. Thus the exchange redistributes the central intrinsic angle defect while leaving the spacetime solution unchanged.

The event can be used in only with . To use , a computational metric and a block-diameter certificate must additionally be supplied. Physical patch size alone is not such a certificate.

A non-equivalent collapsed quadrilateral

Consider

$$A = (-1, 0), \quad B = (1, 0), \quad C = (0, 1 - t), \quad D = (0, t - 1).$$

For $t \neq 1$, the written boundary order *ABCD* has crossing segments *AB* and *CD*. The signed triangle areas are

$$\mathrm{Area}_{\mathrm{sgn}}(A, C, D) = t - 1, \qquad \mathrm{Area}_{\mathrm{sgn}}(B, C, D) = 1 - t.$$

The convex hull has area $2|1 - t|$, which vanishes at $t = 1$; the entire two-dimensional patch collapses to a segment. The proposed replacement *AB* has constant length *2*, rather than being born from zero. Relabelling orientations does not force an exchange of geometric diagonals.

With spacetime coordinate $T = t$, the *A, B* trajectories are timelike and the *C, D* trajectories are null. These causal classifications are not by themselves a prohibition on polygonal coordinates. The failures relevant here are the crossing boundary, the collapse of the whole patch, and the absence of a replacement edge emerging from zero. This quadrilateral is not the certified exchange.

Limitations and successor problems

What the interface does not imply

The schedule results require a prescribed metric, support structure, and channel list. The directional obstruction in prevents replacing causal-region inclusion by a scalar comparison. Periodic graphs can have anisotropic velocity polytopes [Fritz2013]. Random sprinkling is not an automatic cure: the obstruction to intrinsically associating a nontrivial finite-valency graph with a Minkowski sprinkling in a Lorentz-equivariant manner requires separate attention [BombelliHensonSorkin2009]. Likewise, preferred-frame effects under interactions are a separate question; the radiative mechanism studied in [Collins2004] does not follow from, or disappear because of, our

channel identities.

The geometric example has no particle in its moving support. Its internal affine transitions are pure gauge in one flat development. It therefore does not test an event that changes gluing in a region with nontrivial spacetime monodromy.

A deterministic nonlinear state-functional coupling is not automatically a linear classical–quantum evolution. One could instead specify classical–quantum states

$$\sigma = \sum_{g} |g\rangle\langle g| \otimes \rho_g$$

and CP maps $\Psi_{g^{\prime}|g}$ satisfying $\sum_{g^{\prime}} \Psi_{g^{\prime}|g}$ trace preserving for each g, with

$$\mathfrak{F}(\sigma) = \sum_{g,g^{\prime}} |g^{\prime}\rangle\langle g^{\prime}| \otimes \Psi_{g^{\prime}|g}(\rho_g).$$

Such a model would need its own geometric admissibility conditions. It is not an automatic rewriting of .

Approximate spectator or deferred-readout results also require continuity estimates for the coupling and stability margins for the controller. An arbitrarily small change of an input functional can cross a discontinuous guard. The exact theorems contain no hidden small-error estimate.

Capacity and target compatibility

Let $\mathfrak{I}_K$ be the permitted implementations at capacity K, with $\mathfrak{I}_K \subseteq \mathfrak{I}_{K^{\prime}}$ for $K \le K^{\prime}$. For a fixed task error, put

$$\varepsilon_K = \inf_{\mathscr{J} \in \mathfrak{I}_K} \mathrm{Err}(\mathscr{J}), \qquad \inf \varnothing = +\infty.$$

Then $\varepsilon_{K^{\prime}} \le \varepsilon_K$, simply because the infimum is taken over a larger set. There is no implication that $\varepsilon_K \to 0$.

Feasibility, approximation of a specified target, and compatibility of two ideal targets are separate questions. The last depends on the named observation algebra. The residual obstruction in cannot be cured merely by allocating more work bits.

Successor problems

Construct a certified transition whose support contains a particle or nontrivial spacetime monodromy. Supply complete affected labels, boundary matching, ordered affine products, location transport, control assumptions, and a decoder equation. Approximate event detection must distinguish geometric error from event-time and event-resolution error.

For a specified observation algebra and resource model, construct nested implementation families with an explicit approximation rate. Separate target incompatibility from implementation error.

For a moving geometric model and a computational substrate, prove directional inclusion of physical dependency regions in scheduled regions. A scalar speed bound alone is insufficient.

After a renormalized composite has been independently constructed on all compact tests, specify its accessible observable structure and test descent of its mandatory presentations through the declared counterterm equivalence. Establish any required products separately; do not assume multiplication on a quotient of distributions.

These problems are successors, not missing premises of the finite interface theorems. In particular, the present results do not settle the interacting-product problem of the separate Wick track.

Conclusion

A destructive measurement primitive fixes an output signature, not a geometric observation policy. The latter must declare coupling observables, control, resource checks, locations, and accessible residual outputs.

With those declarations in place, arbitrary spectator extensions preserve geometry and accessible reductions under spectator-blind control. Restricted coherent deferral reproduces destructive readout on the common record–matter interface. Record compatibility is exactly an effect pullback, with an exact unassisted event-norm error. Residual compatibility is stronger: for sharp readouts and a coarse Lüders instrument, its remaining obstruction is

intra-outcome coherence. Continuations preserve observation boundaries precisely when their adjoints preserve the declared observable spaces.

The four-face certificate realizes one discrete geometric event on a prescribed slicing of a fixed spacetime, including its boundary, guard, transport, and finite reversible code. It neither derives geometry from measurement nor changes the ambient gravitational solution.

The observation-boundary problem is closed within the stated finite model. The distributional graph problem and gravitational dynamics remain separate tasks.

appendix

Intrinsic angle defect at the exchange

The angle at a vertex of a spacelike planar face is computed using the positive definite restriction of η to that face. At $s = 0$, the two edge rays of the E-face may be represented by

$$u_{\pm} = (p, 1, \pm p/q).$$

Their squared norms are equal to

$$N_E^2 = 1 - p^2 + (p/q)^2,$$

and their inner product is $1 - p^2 - (p/q)^2$. Thus

$$\cos \alpha_E = \frac{1 - p^2 - (p/q)^2}{1 - p^2 + (p/q)^2} = -\frac{1121}{1921}.$$

Reflection gives the same angle for W.

For the N-face, use

$$v_{\pm} = (q, \pm q/p, 1).$$

Then

$$\cos \alpha_N = \frac{1 - q^2 - (q/p)^2}{1 - q^2 + (q/p)^2} = \frac{671}{1921},$$

and S has the same angle. Both angles lie in $(0, \pi)$.

The event junction therefore has defect

$$\kappa_0 = 2\pi - 2\alpha_E - 2\alpha_N.$$

Since

$$\cos \alpha_N = \frac{671}{1921} < \frac{1121}{1921} = \cos (\pi - \alpha_E),$$

strict decrease of cosine on $[0, \pi]$ gives $\alpha_N > \pi - \alpha_E$. Hence $\kappa_0 < 0$.

At the incoming northern triple junction, the central edge points in direction $d = (0, 0, -1)$. Its angle β_E with the north-east arm satisfies

$$\cos \beta_E = -\frac{p/q}{\sqrt{1 - p^2 + (p/q)^2}} = -\sin (\alpha_E/2).$$

Because $\beta_E \in (0, \pi)$,

$$\beta_E = \frac{\pi + \alpha_E}{2}.$$

The W-angle is the same, and the N-angle is α_N. The triple-junction defect is consequently

$$2\pi - 2\beta_E - \alpha_N = \pi - \alpha_E - \alpha_N = \frac{\kappa_0}{2}.$$

Reflection gives the same value at the southern incoming junction.

At an outgoing triple junction the roles of E and N exchange: the two equal angles are $\beta_N = (\pi + \alpha_N)/2$, and the

third is α_E. Its defect is again $\kappa_{0/2}$. All relevant edge directions are constant within their respective epochs, so these defects are independent of s there.

Thus the sum of the two central triple defects equals the single event defect on both sides. This is a statement about intrinsic geometry of the marked piecewise-flat slice. It is consistent with the identity ambient transition products proved in .

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