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# Born–Lüders as the unique measurement strategy of a strictly linear quantum λ-calculus

*A self-contained finite characterization in the Enter calculus*

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## Abstract

We characterize the destructive computational-basis measurement primitive of a fixed strictly linear quantum lambda calculus. The admissible strategies are state sufficient, have normalized outcome laws, and assign pure conditional residuals to positive-probability branches. The argument separates two logical layers. First, Born probabilities and destructive projective update define a concrete calculus with type safety, a syntactic linear step bound, observational confluence, circuit adequacy, semantic no-cloning, and internal no-signaling. Second, only the measurement transition is left unknown. Composability, no-signaling, nullity, continuity, and a typed record equation are satisfied by the Born strategy and uniquely determine it within the stated class. Two independent finite constructions establish diagonal anchoring without the record equation. Whole-register records then determine coherent outcome laws, composability determines partial readout, and branch-specific continuation tests determine residual rays. We also prove resource-budgeted stability estimates and derive mixed-state, adaptive, and finite-instrument consequences. A static appendix constructs positive non-Born marginal assignments and exhibits their failure of operational nullity. Later record, decoder, instrument, and cutoff interfaces are kept separate from the premises of primitive uniqueness.



## Bibliographic record

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par **Scope** par The result is conditional on the structural class stated below. It is not a reconstruction of qubit kinematics, an independence theorem for the five requirements, or a proof of record-free uniqueness. It is not a physical Frauchiger–Renner no-go theorem. No field-theoretic instrument, infinite apparatus, microscopic evaporation channel, horizon construction, characteristic quantum coefficient, or energy-uniform channel approximation is derived. par

## Introduction and logical scope

Fix a quantum programming language: its types, allocation, tensor composition, coherent gates, and classical control. Leave only the outcome law and conditional update of a destructive measurement primitive unspecified. Which strategies interpret that primitive while satisfying specified equations between experiments?

The primitive considered here is *Enter*, destructive computational-basis readout of a finite qubit register. A configuration contains one global pure state. Runtime qubit names are access rights to its tensor factors; they are not individual pure-state assignments. Readout consumes the measured names, returns classical bits, and leaves a

state on the surviving factors.

This manuscript gives a self-contained, scope-restricted presentation of the finite Enter-calculus characterization of Cheung [EnterCalculus]. The central result is Theorem . Here ``Born–Lüders" means projective conditioning followed by removal of the measured factors, not retention of those factors. The distinction between outcome probabilities and state change is essential; the historical projective update is discussed in Lüders' work and its English translation [LudersTranslation].

### Two logical layers

**Layer I: specified semantics.** The concrete calculus *EC_Born* has Born probabilities and destructive projective update by definition. Safety, termination, confluence, adequacy, and no-signaling are theorems about that already specified semantics.

**Layer II: unknown measurement.** A strategy $S = (P, U)$ replaces only the measurement transition. The structural class fixes the domain of the characterization. The five requirements

$$(C)^{\star}, \qquad (NS)^{\star}, \qquad (N)^{\star}, \qquad (K)^{\star}, \qquad (R)^{\star}$$

are conditions on the unknown strategy.

The two directions are

$$B \vDash T, \qquad S \vDash T \Rightarrow S = B.$$

The first establishes inhabitation; the second establishes uniqueness within the declared class. A concrete Born calculation is not available for an unknown strategy merely because both strategies interpret the same syntax.

### Assumption ledger

toprule Declared, not derived Finite-dimensional qubit kinematics, ordinary tensor composition, global pure states, coherent gates, state sufficiency, normalized outcome laws, pure positive-branch residuals, the structural conditions of Assumption , and the five requirements of Definition .

addlinespace Checked syntactically The record transformation is definable and well typed using allocation and CNOT. Every quantum spectator is consumed or returned.

addlinespace Derived The metatheory of the specified calculus; diagonal anchoring from the first four requirements; and, using records, Born laws and destructive projective residual rays for every admissible strategy.

bottomrule

No silent quantum deletion is added to the language. Algebraic partial traces used in proofs are not term-level weakening operations.

### Relation to other approaches

Quantum lambda calculi with classical control already provide well-developed operational and typing frameworks; the affine calculus of Selinger and Valiron is a representative example [SelingerValiron]. The present language uses strict, branch-wise linear ownership. Its characterization question concerns a variable measurement transition in a fixed language, rather than type safety alone.

Gleason's theorem concerns measures on Hilbert-space subspaces subject to a different assumption package [Gleason]. The present proof does not begin with probabilities for all effects or projectors. It characterizes one fixed-basis primitive; other measurements are represented later by programs.

The uniform-correlation argument uses a system/environment permutation mechanism related to envariance arguments [ZurekEnvariance]. Its premises are stated explicitly as operational requirements. The importance of distinguishing such premises from derived probabilities is emphasized in the analysis of Schlosshauer and Fine [SchlosshauerFine]. Neither envariance nor a separate noncontextuality axiom is imported as an additional premise.

Finally, the companion manuscripts discussed in Section concern implementation and observational boundaries. They are not premises of Theorem .

## Language, typing, and ownership

### Types and terms

Types are

$$A, B ::= 1 \mid \mathrm{Bit} \mid \mathrm{Q} \mid A \multimap B \mid A \otimes B.$$

Tensor powers are right nested: $A^{\otimes 0} = 1$ and $A^{\otimes 1} = A$. The empty tuple is $*$. Linear pairing and elimination implement the usual associativity and unit isomorphisms. Tuple notation suppresses these administrative maps, not resource uses.

Terms are

$$\begin{aligned} t, u ::= {} & x \mid q \mid * \mid 0 \mid 1 \mid \lambda x : A.\, t \mid t\, u \mid \langle t, u \rangle \\ & \mid \mathrm{let}\, \langle x, y \rangle = t \,\mathrm{in}\, u \mid \mathrm{let}\, * = t \,\mathrm{in}\, u \mid \mathrm{if}\ t \,\mathrm{then}\, u \,\mathrm{else}\, u' \\ & \mid \mathrm{new} \mid V(t) \mid \mathrm{Enter}(t). \end{aligned}$$

Source variables are $x, y, \ldots$. Runtime names $q, r, \ldots$ are introduced by allocation. Source programs contain no runtime names. Every finite-register unitary is available as a gate; a dense-gate variant is given in Corollary .

We abbreviate $(\lambda x{:}\, A.\, u)t$ by $\mathit{let}\ x = t\ \mathit{in}\ u$. Tuple-pattern binders abbreviate iterated linear tuple elimination.

### Typing

Contexts assign types to distinct source variables and runtime names. Contexts separated in a multiplicative rule are disjoint. There is no structural weakening or contraction.

$$\frac{}{x : A \vdash x : A} \qquad \frac{}{q : \mathrm{Q} \vdash q : \mathrm{Q}} \qquad \frac{}{\vdash * : 1} \qquad \frac{}{\vdash 0 : \mathrm{Bit}} \qquad \frac{}{\vdash 1 : \mathrm{Bit}}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\, t : A \multimap B} \qquad \frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\, u : B}$$

$$\frac{\Gamma \vdash t : A \quad \Delta \vdash u : B}{\Gamma, \Delta \vdash \langle t, u \rangle : A \otimes B}$$

$$\frac{\Gamma \vdash t : A \otimes B \quad \Delta, x : A, y : B \vdash u : C}{\Gamma, \Delta \vdash \mathrm{let}\, \langle x, y \rangle = t \,\mathrm{in}\, u : C}$$

$$\frac{\Gamma \vdash t : 1 \quad \Delta \vdash u : C}{\Gamma, \Delta \vdash \mathrm{let}\, * = t \,\mathrm{in}\, u : C}$$

$$\frac{\Gamma \vdash b : \mathrm{Bit} \quad \Delta \vdash u : A \quad \Delta \vdash u' : A}{\Gamma, \Delta \vdash \mathrm{if}\ b \,\mathrm{then}\, u \,\mathrm{else}\, u' : A}$$

$$\frac{}{\vdash \mathrm{new} : \mathrm{Q}} \qquad \frac{\Gamma \vdash t : \mathrm{Q}^{\otimes n}}{\Gamma \vdash V(t) : \mathrm{Q}^{\otimes n}} \qquad \frac{\Gamma \vdash t : \mathrm{Q}^{\otimes n}}{\Gamma \vdash \mathrm{Enter}(t) : \mathrm{Bit}^{\otimes n}}.$$

The conditional is additive: its alternatives use the same context, but only one alternative executes. The bit *1* selects the then-branch and *0* the else-branch.

### Branch-wise linearity

**Definition.** An occurrence selection retains the guard of each conditional and exactly one of its alternatives, recursively. It is a selection in the syntax tree, not a reduction. The set *own (t)* consists of the free runtime names of $t$.

**Lemma.** If $\Gamma \vdash t : A$, every resource in $\Gamma$ has exactly one free occurrence in every occurrence selection. The free runtime names of $t$ are exactly the runtime names in $\Gamma$, and the two alternatives of a conditional have identical free-runtime-name sets.

*Proof.* Induct on typing. Multiplicative rules partition contexts. Abstraction removes its bound variable from the free context. For a conditional, the guard uses its own context, and either selected alternative uses the shared alternative context once. The same induction proves equality of the alternatives' name sets.

A resource may therefore appear in both unexecuted alternatives without being duplicated during execution.

Classical copying and deletion are definable:

$$\begin{aligned}\mathrm{copyBit} &= \lambda b : \mathrm{Bit}.\ \mathrm{if}\ b\ \mathrm{then}\ \langle 1,1\rangle\ \mathrm{else}\ \langle 0,0\rangle,\\ \mathrm{dropBit} &= \lambda b : \mathrm{Bit}.\ \mathrm{if}\ b\ \mathrm{then}\ \ \mathrm{else}\ .\end{aligned}$$

Their types are *Bitmultimap Bit* ⊗ *Bit* and *Bitmultimap 1*. Finite classical functions are definable by finite case analysis.

**Remark.** The type *Qmultimap Q* ⊗ *Q* is inhabited:

$$\lambda q : \mathrm{Q}.\ \langle q, \mathrm{new}\rangle$$

has that type. Typing prohibits duplicate ownership, not all terms with a possible cloning type. Universal cloning is excluded semantically in Theorem . Quantum consumption by an explicit measurement followed by classical deletion is also different from structural quantum weakening.

## Configurations and specified Born semantics

### One global state

For a finite live-name set $L$, write

$$\mathscr{H}_L = \mathrm{bigotimes}_{q \in L}\, \mathbb{C}^2.$$

An ordering is chosen for matrices and bit strings. Changing the ordering simultaneously permutes names and tensor factors.

For $S \subseteq L$, let $\bar{S} = L \setminus S$, and define

$$P_\kappa^S = |\kappa\rangle\langle\kappa|_S \otimes I_{\bar{S}}, \qquad M_\kappa^S = \langle\kappa|_S \otimes I_{\bar{S}}.$$

The projector acts on $\mathscr{H}_L$, whereas

$$M_\kappa^S : \mathscr{H}_L \to \mathscr{H}_{\bar{S}}$$

removes the measured factors. It satisfies

$$(M_\kappa^S)^\dagger M_\kappa^S = P_\kappa^S.$$

**Definition.** A configuration is *[ψ, t]*, where $\psi$ is a unit vector in $\mathscr{H}_L$, modulo global phase, and *own (t) = L*. It is well typed at $A$ if

$$(q : \mathrm{Q})_{q \in L} \mathrm{vdash}\, t : A.$$

Allocation extends the name-to-factor ordering by a fresh name; destructive readout restricts it to surviving names.

An entangled name is not assigned an individual pure state. All live names refer to factors of the same vector.

### Values and evaluation

Values are

$$v ::= * \mid 0 \mid 1 \mid q \mid \lambda x : A.\ t \mid \langle v_1, v_2\rangle.$$

Left-to-right call-by-value contexts are

$$\begin{aligned}E ::= \quad & [\,] \mid E\,t \mid v\,E \mid \langle E, t\rangle \mid \langle v, E\rangle\\ & \mid \mathrm{let}\,\langle x,y\rangle = E\ \mathrm{in}\ t \mid \mathrm{let}\ = E\ \mathrm{in}\ t\\ & \mid \mathrm{if}\ E\ \mathrm{then}\ u\ \mathrm{else}\ u' \mid V(E) \mid \mathrm{Enter}(E).\end{aligned}$$

They do not enter abstraction bodies, let-bodies, or conditional alternatives.

Administrative root reductions are

$$\begin{aligned}(\lambda x : A.\ t)v &\to t[v/x],\\ \mathrm{let}\,\langle x,y\rangle = \langle v,w\rangle\ \mathrm{in}\ t &\to t[v/x, w/y],\\ \mathrm{let}\ = *\ \mathrm{in}\ t &\to t,\\ \mathrm{if}\ 1\ \mathrm{then}\ u\ \mathrm{else}\ u' &\to u,\\ \mathrm{if}\ 0\ \mathrm{then}\ u\ \mathrm{else}\ u' &\to u'.\end{aligned}$$

They preserve the state and are closed under $E$. Together with allocation and gates,

$$[\psi, E[\mathrm{new}]] \to_{\mathrm{d}} [\psi \otimes |0\rangle_r, E[r]] \quad (r\text{fresh}),$$

$$[\psi, E[V(\vec{q}_S)]] \to_{\mathrm{d}} [V_S\psi, E[\vec{q}_S]],$$

they form the deterministic transitions.

The specified Born measurement transition is

$$\mathrm{boxed}\,[\psi, E[\mathrm{Enter}(\vec{q}_S)]] \to_{\mathrm{B}} \left[\frac{M_\kappa^S\psi}{\sqrt{b_\kappa}}, E[\vec{\kappa}]\right] \quad \text{with probability} \quad b_\kappa = \left\|M_\kappa^S\psi\right\|^2 > 0.$$

There is no transition on a zero-weight branch. Measured names are removed. For $S = \varnothing$, the outcome is * and the state is unchanged. A zero-qubit gate changes only an irrelevant phase.

The calculus with these transitions is *EC_Born*. Equation is a definition in Layer I.

**Safety**

**Lemma.** If

$$\Gamma, x : A\mathrm{vdash}\, t : B, \qquad \Delta\mathrm{vdash}\, v : A$$

with disjoint contexts, then

$$\Gamma, \Delta\mathrm{vdash}\, t[v/x] : B.$$

In a context consisting only of runtime names, values of types *Q*, *Bit*, *1*, tensor, and function are respectively names, bits, *, pairs, and abstractions.

*Proof.* Induct on the typing derivation for substitution. A multiplicative rule places $x$ in one premise. In a conditional, $x$ occurs either in the guard context or in the shared alternative context. In the latter case substitute in both alternatives and reuse the same inserted context $\Delta$ in the additive rule. Canonical forms follow from the value grammar and possible final typing rules.

**Theorem.** Every deterministic or positive-probability Born transition preserves the type of a well-typed configuration. Every nonvalue well-typed configuration has an enabled transition.

*Proof.* Administrative preservation follows from substitution. Conditional selection retains the shared alternative resources by Lemma . Allocation adds one name and one factor. A gate changes neither ownership nor type. Readout removes precisely its argument names and returns a bit tuple of the required type.

For progress, follow the call-by-value typing decomposition until a root operation is reached. Canonical forms supply the required value shape. Gate and readout arguments are tuples of distinct runtime names, or * at arity zero. Finally,

$$\sum_\kappa \left\|M_\kappa^S\psi\right\|^2 = 1$$

provides at least one positive measurement branch.

**A syntactic linear step bound**

Define

$$m(x) = m(q) = m(*) = m(0) = m(1) = 0, \qquad m(\lambda x.\, t) = m(t),$$

$$m(\langle t, u\rangle) = m(t) + m(u), \qquad m(\mathrm{new}) = 1,$$

$$m(V(t)) = m(\mathrm{Enter}(t)) = 1 + m(t),$$

and

$$\begin{aligned} m(tu) &= 1 + m(t) + m(u),\\ m(\mathrm{let}\,\langle x, y\rangle = t \text{ in } u) &= 1 + m(t) + m(u),\\ m(\mathrm{let}\ = t \text{ in } u) &= 1 + m(t) + m(u),\\ m(\mathrm{if}\ b \text{ then } u \text{ else } u') &= 1 + m(b) + \max\{m(u), m(u')\}. \end{aligned}$$

**Lemma.** Under the hypotheses of Lemma ,

$$m(t[v/x]) \le m(t) + m(v).$$

Simultaneous substitutions for disjoint linear resources increase *m* by at most the sum of the inserted measures.

*Proof.* Induct on typing, simultaneously proving the multiple-substitution statement. Multiplicative premises receive disjoint substitutions. If substitutions of total measure *a* enter the shared alternative context, then

$$\max\{m(u[\cdot]), m(u'[\cdot])\} \le \max\{m(u), m(u')\} + a.$$

Substitutions in the guard are disjoint from these. This proves the conditional case; the other cases are additive.

**Theorem.** Every reduction sequence from a well-typed configuration *[ψ, t]* has length at most *m(t)*.

*Proof.* Every root transition decreases *m* by at least one. Beta reduction and tuple elimination use Lemma ; conditional selection uses the maximum. Other transitions remove one counted operation. Along an evaluation-context hole the measure is additive, since the hole is never inside a body or unselected alternative. The same decrease therefore holds contextually.

The bound counts primitive syntax operations. A unitary is one node regardless of its matrix dimension or synthesis cost. Since *m(t)* is bounded by the number of written operation nodes, this is a syntactically linear bound, not a physical complexity claim.

**Definition.** For an output of type *Bit^{⊗m} ⊗ Q^{⊗k}*, *Obs [ψ, t]* is the terminal law of the classical tuple. The quantum tuple remains returned and owned; this observation does not test its state. Purely classical output is included using the unit isomorphism.

Termination, finite branching, and normalized transitions make these laws well defined.

## Metatheory of the specified calculus

### Liberal scheduling and observational confluence

Liberal contexts permit independent argument evaluation:

$$\begin{aligned} L ::= {} & [\,] \mid L\,t \mid t\,L \mid \langle L, t\rangle \mid \langle t, L\rangle \\ & \mid \mathrm{let}\,\langle x, y\rangle = L \text{ in } u \mid \mathrm{let}\ _ = L \text{ in } u \\ & \mid \mathrm{if}\ L \text{ then } u \text{ else } u' \mid V(L) \mid \mathrm{Enter}(L). \end{aligned}$$

They still do not enter bodies or alternatives. A scheduler chooses an enabled redex at every nonterminal configuration. Configurations are identified up to consistent live-name renaming, tensor permutation, and global phase.

**Lemma.** Distinct enabled liberal redexes occupy nonnested positions and own disjoint runtime-name sets. Firing either and then the residual of the other gives equal joint weights and equivalent resulting configurations.

*Proof.* An enabled root redex has no enabled accessible redex inside it: its required arguments are values, and bodies and alternatives are inaccessible. Distinct redexes are therefore nonnested. Their positions are separated by context-splitting rules. Shared additive contexts occur only in inaccessible alternatives.

Disjoint administrative transitions commute. Allocations commute up to fresh-name renaming and commute with operations on existing factors. Disjoint gates commute. A gate commutes with destructive readout on disjoint factors, with the canonical identification of the surviving output space.

For disjoint readouts,

$$M_\lambda^T M_\kappa^S = M_{\kappa\lambda}^{S\cup T} = M_\kappa^S M_\lambda^T.$$

Both orders have joint weight

$$\left\| M_{\kappa\lambda}^{S\cup T}\psi \right\|^2$$

and, on a positive joint branch, the same normalized residual. Zero joint weights contribute zero in both orders.

**Theorem.** All liberal schedulers yield the same terminal observation law for a well-typed *EC_Born* configuration.

*Proof.* The termination measure also decreases in liberal contexts. Induct on *m(t)*. If two first choices differ, refine each successor by firing the residual of the other redex. Lemma matches the resulting configurations with equal weights. The induction hypothesis identifies their terminal laws. If the first choices agree, apply induction to their

successors.

**Lemma.** Joint readout of disjoint *S, T* and successive readout in either order have the same ordered joint law, with all other names returned.

*Proof.* Every joint weight equals ‖*M_{κλ}^{S ∪ T}ψ*‖^2.

Confluence compares evaluations of one program. Composability compares distinct programs.

### Path maps and circuit adequacy

A *first-order interface* has only finite classical and quantum data at input and output. Internal higher-order terms are allowed. For a fixed classical input, the quantum input and output spaces are fixed by the interface.

**Lemma.** For a fixed well-typed program with a first-order interface, every complete classical history *h* determines a linear map *K_h* from its quantum input space to its returned quantum output space. It is a composition of allocation isometries, gates, wiring permutations, and destructive readout maps. For a pure input $\psi$,

$$\Pr(h) = \|K_h\psi\|^2, \qquad \psi_h = \frac{K_h\psi}{\|K_h\psi\|} \quad \text{when} \Pr(h) > 0,$$

and

$$\sum_h K_h^\dagger K_h = I.$$

*Proof.* A complete history fixes all classical choices. Administrative reductions determine syntax and wiring but do not inspect the quantum vector. Multiplying the quantum maps along the history gives *K_h*. The successive conditional normalizations cancel in the product of branch probabilities, leaving ‖*K_hψ*‖^2.

For completeness use the finite formal outcome tree, including branches whose weight is zero on a particular input. At a readout,

$$\sum_\kappa (M_\kappa^S)^\dagger M_\kappa^S = I.$$

Allocation is isometric and gates are unitary. Backward induction on the tree proves completeness. All paths have the same output Hilbert space after the canonical identification by the returned tuple.

**Theorem.** Every well-typed program with a first-order interface denotes a finite quantum instrument. Conversely, every finite circuit built from the available gates, zero-state allocation, destructive computational-basis readout, and finite classical control has a well-typed implementation with the same instrument.

*Proof.* For reported output *y*, define

$$\Phi_y(\rho) = \sum_{h\,:\,\mathrm{out}(h) = y} K_h \rho K_h^\dagger.$$

The maps are completely positive and their sum is trace preserving by Lemma . Unrolling the finite formal evaluation tree gives a finite adaptive circuit.

Conversely, input wires are linear arguments. Fresh wires use *new*, routing uses pairing and tuple elimination, and operations use gates or *Enter*. Finite control uses conditionals, *copyBit*, and *dropBit*. Every quantum wire is passed to exactly one next operation, explicitly consumed, or returned. Induction through the circuit identifies its path maps.

The CP-map language here is the standard finite-dimensional instrument language; its operator-sum and dilation background is provided by Choi and Stinespring [Choi,Stinespring]. The adequacy proof itself is included above.

**Theorem.** No closed term *c: Qmultimap Q ⊗ Q* returns two copies of every pure qubit input.

*Proof.* By Lemma , its averaged output map $\Phi$ is linear on density operators. A universal cloner would satisfy

$$\Phi(|\varphi\rangle\langle\varphi|) = |\varphi\rangle\langle\varphi|^{\otimes 2}.$$

But

$$I/2 = \tfrac{1}{2}(\mathrm{ketbra}\,00 + \mathrm{ketbra}\,11) = \tfrac{1}{2}(\mathrm{ketbra}\,{+}{+} + \mathrm{ketbra}\,{-}{-}).$$

Linearity gives two expressions for *Φ(I/2)*. Their diagonal entries at $|01\rangle$ are respectively *0* and *1/4*, a

contradiction.

This is a calculus-level application of semantic no-cloning, rather than an inference from the absence of contraction; compare Wootters and Zurek [WoottersZurek].

### Internal no-signaling

A bipartite experiment has a state on *Asqcup B* and two local programs owning disjoint initial name sets. They may allocate locally but exchange neither names nor messages. An admissible interleaving retains each local program's left-to-right order.

**Theorem.** Alice's terminal classical law is independent of Bob's local program and of the admissible interleaving.

*Proof.* Let *K_a* and *L_b* be their local path maps, including allocations and returned outputs. Disjoint maps commute with the canonical output identifications. The joint history weight is *‖(K_a ⊗ L_b)ψ‖^2*. Consequently

$$\begin{aligned}\Pr(a) &= \sum_b \langle\psi|K_a^\dagger K_a \otimes L_b^\dagger L_b|\psi\rangle \\ &= \langle\psi|K_a^\dagger K_a \otimes I|\psi\rangle \\ &= \operatorname{tr}(K_a^\dagger K_a \rho_A),\end{aligned}$$

where the second line uses local completeness. Summing histories with the same reported output proves the claim. The calculation is independent of the interleaving.

This concerns unconditioned local records. Conditional dependence on a reported remote outcome is not excluded.

## Typed records and experiment interfaces

### The record transformation

For *k* source and *k* target qubits, let *C_k* be the grouped CNOT unitary

$$C_k(|z\rangle_S|w\rangle_R) = |z\rangle_S|w \oplus z\rangle_R.$$

It is a tensor product of CNOTs together with the required linear regrouping maps. At *k = 0*, it is the identity.

**Definition.** Let *L = Ssqcup B*, *|S| = k*, and suppose

$$(q : \mathrm{Q})_{q \in B} \mathrm{vdash}\, u : \mathrm{Bit}^{\otimes m}.$$

The source term is

$$t_{S,u} = \mathrm{let}\, x = \mathrm{Enter}(\vec{q}_S)\ \mathrm{in}\ \langle u, x\rangle, \qquad x \notin u.$$

Its deferred term is

$$\begin{aligned}\mathrm{defer}_S(t_{S,u}) = \quad & \mathrm{let}\, \vec{r} = \mathrm{new}^k\ \mathrm{in} \\ & \mathrm{let}\, \langle \vec{s}, \vec{r}' \rangle = C_k(\langle \vec{q}_S, \vec{r} \rangle)\ \mathrm{in} \\ & \big\langle \big\langle u, \mathrm{Enter}(\vec{r}') \big\rangle, \vec{s} \big\rangle.\end{aligned}$$

Here *new^k* denotes a tuple of *k* fresh allocations. The continuation consumes the entire register *B*. It may contain allocations, gates, readouts, and classical control, but uses neither source nor record names.

After the stipulated tuple isomorphisms, the source has classical arity *m + k*, while the target has type

$$\mathrm{Bit}^{\otimes(m+k)} \otimes \mathrm{Q}^{\otimes k}.$$

Both report the same ordered classical pair *(c, x)*. The target also returns the original source register. Left-to-right evaluation completes *u* before reading the record.

Every retained source name occurs once in the returned tuple. Thus neither term duplicates a quantum name or silently drops a spectator. In particular, *u = ** is legal only when *B = ∅*.

**Proposition.** In *EC_Born*,

$$\mathrm{Obs}_B[\psi, t_{S,u}] = \mathrm{Obs}_B[\psi, \mathrm{defer}_S(t_{S,u})].$$

*Proof.* Write

$$\psi = \sum_{\kappa} |\kappa\rangle_S |\xi_\kappa\rangle_B.$$

For a path $h$ of $u$, let $K_h$ be its path map. The source's joint weight for $h$, $\kappa$ is $\|K_h\xi_{\kappa}\|^2$.

Allocation and copying in the target produce

$$\sum_{\kappa} |\kappa\rangle_S |\xi_\kappa\rangle_B |\kappa\rangle_R.$$

After path $h$, the unnormalized vector is

$$\sum_{\kappa} |\kappa\rangle_S (K_h\xi_\kappa) |\kappa\rangle_R.$$

Record readout has the same joint weight $\|K_h\xi_{\kappa}\|^2$. Sum histories with the same reported continuation output.

The proposition uses the specified Born semantics. The same equation becomes a requirement on unknown strategies below; its two typed terms remain unchanged.

**Spectators and local experiments**

For disjoint $S, T \subseteq L$, put $Z = L \setminus (S \cup T)$. The sequential experiment is

$$\text{let}\, x = \text{Enter}(\vec{q}_S)\ \text{in?let}\, y = \text{Enter}(\vec{q}_T)\ \text{in?}\langle\langle x, y\rangle, \vec{q}_Z\rangle.$$

The reverse experiment reads $T$ first but reports the same ordered classical pair. Joint readout of $S \cup T$ likewise returns $Z$. These terms have the same observation interface after linear tuple rearrangement.

A local identity experiment may return its names with a constant classical unit. A remote allocation experiment returns the additional allocated name. Thus ancilla comparisons below do not use quantum weakening.

There are two record scopes:

• *readout instances*: $S = L$ and $u = *$;

• *continuation instances*: $u$ consumes all of $L \setminus S$. The outcome proof uses only readout instances. The update proof uses

$$u = \text{Enter}(V(\vec{q}_{L \setminus S})).$$

No equation for arbitrary surrounding contexts is assumed.

## Unknown strategies and their requirements

**Assumption.** Only the measurement transition varies. The admissible class has the following properties.

• **Qubit kinematics.** For every finite live-name set $L$, states are unit rays in $\mathscr{H}_L$, with ordinary tensor composition.

• **Fixed coherent dynamics.** All finite-register unitaries and fresh $|0\rangle$ allocation act as in Layer I, with their stated linear-algebraic composition and tensor placement.

• **State sufficiency.** The outcome law $P_{\psi}^S$ and positive-branch residual ray $U_{\psi}^S(\kappa)$ depend only on the current global ray and measured register, not on history, apparatus identity, or hidden memory.

• **Normalized laws and pure residuals.** $P_{\psi}^S$ is a probability distribution on $\{0, 1\}^S$. Each positive-probability outcome has a unit residual ray on $\mathscr{H}_{L \setminus S}$. Values on zero-probability outcomes are not part of the strategy.

• **Permutation equivariance.** Consistent relabeling of names, tensor factors, measured positions, and outcome coordinates transports laws and residuals accordingly.

• **Unbounded finite resources.** The strategy is defined at every finite size. Experiments may use any finite number of ancillas. Witnesses need not have the size of the system whose law is being characterized.

• **Evaluation convention.** Single programs use left-to-right evaluation. Bipartite experiments interleave their local left-to-right executions. General liberal scheduler independence is not assumed.

No closure under silent quantum deletion is included. Arbitrary input rays are admissible configurations, whether or not a restricted gate language can prepare them exactly.

**Definition.** A strategy is a pair $S = (P, U)$ in the structural class, interpreting readout by

$$[\psi, E[\mathrm{Enter}(\vec{q}_S)]] \to_S [\mathrm{U}^S_\psi(\kappa), E[\vec{\kappa}]] \quad \text{with probability} \mathrm{P}^S_\psi(\kappa) > 0.$$

Safety and the step bound use only types, ownership, and normalized branching, so terminal laws also exist for unknown strategies. The concrete diamond, path maps, and no-signaling theorem are not transferred before characterization.

**Definition.** The theory $T$ consists of:

**\((\mathrm C)^\star\): composability.** For disjoint $S, T \subseteq L$, successive readout in either order and joint readout of $S \cup T$ yield the same ordered joint law, with all other names returned. On a positive first branch,

$$\mathrm{P}^{S\cup T}_\psi(\kappa, \lambda) = \mathrm{P}^S_\psi(\kappa)\mathrm{P}^T_{\mathrm{U}^S_\psi(\kappa)}(\lambda).$$

A zero first weight gives zero joint weight. In particular,

$$\mathrm{P}^S_\psi = \mathrm{marg}_S \mathrm{P}^{S\cup T}_\psi.$$

Only the classical joint laws, not complete residual instruments, are equated.

**\((\mathrm{NS})^\star\): no-signaling.** For every bipartition, the terminal classical law of either local program is independent of the other program and of the admissible interleaving. Local programs may allocate, apply gates, read out, use classical control, and return quantum spectators. They exchange neither names nor messages.

**\((\mathrm N)^\star\): nullity.** For every $\psi$, $S$, $\kappa$,

$$M^S_\kappa \psi = 0 \quad \Rightarrow \quad \mathrm{P}^S_\psi(\kappa) = 0.$$

The experiment reads $S$ and returns its complement. This is a zero-support condition, not a vacuum condition or a Born probability formula.

**\((\mathrm K)^\star\): continuity.** At every fixed finite live register and measured subset,

$$[\psi] \mathrm{longmapsto}\, \mathrm{P}^S_\psi$$

is continuous from ray space to the finite probability simplex, equivalently in total variation. No continuity of $U$ is assumed.

**\((\mathrm R)^\star\): records.** For every configuration of Definition ,

$$\mathrm{Obs}_S[\psi, t_{S,u}] = \mathrm{Obs}_S[\psi, \mathrm{defer}_S(t_{S,u})].$$

The continuation consumes the entire complementary register and does not use the first reported outcome.

Each comparison has an explicit typed experiment. The requirements do not assume probability covariance under arbitrary computational-basis permutations; the needed symmetry is derived below.

## Inhabitation and characterization architecture

Define the Born strategy by

$$\mathrm{P}^S_\psi(\kappa) = \left\| M^S_\kappa \psi \right\|^2, \qquad \mathrm{U}^S_\psi(\kappa) = \left[ \frac{M^S_\kappa \psi}{\left\| M^S_\kappa \psi \right\|} \right]$$

on positive branches.

**Theorem.** The Born strategy belongs to the structural class and satisfies *(C)^{⋆}, (NS)^{⋆}, (N)^{⋆}, (K)^{⋆}, (R)^{⋆}*.

*Proof.* The formulas give normalized laws and pure residual rays, are state sufficient and phase invariant, and are equivariant under consistent name and tensor permutation. They are defined at every finite size with the fixed coherent dynamics and evaluation convention.

Composability is Lemma . No-signaling is Theorem . A zero destructive component has zero squared norm, proving nullity. Each squared norm is continuous, and there are finitely many outcomes, proving continuity. Proposition proves records.

**Theorem.** A strategy in Assumption satisfies Definition if and only if it is the Born strategy. For every finite configuration, measured subset $S$, and outcome $\kappa$,

$$\mathrm{P}^{S}_{\psi}(\kappa) = \left\| M^{S}_{\kappa}\psi \right\|^{2}.$$

Whenever this number is positive,

$$\mathrm{U}^{S}_{\psi}(\kappa)\mathrm{doteq}\,\frac{M^{S}_{\kappa}\psi}{\left\| M^{S}_{\kappa}\psi \right\|},$$

where *doteq* denotes equality up to global phase. No residual is specified on a zero-probability branch.

The uniqueness proof occupies Sections –. Its dependencies are: toprule Step Dependencies

midrule Remote invariance and locality *(NS)^{⋆}*, state sufficiency, fixed coherent dynamics, and purification linear algebra.

Uniform correlations Remote invariance, *(C)^{⋆}*, *(N)^{⋆}*.

Symmetry anchoring Locality, *(C)^{⋆}*, *(N)^{⋆}*, *(K)^{⋆}*, and (S5).

Fine-graining anchoring Locality, uniform correlations, *(C)^{⋆}*, *(N)^{⋆}*, *(K)^{⋆}*.

Whole-register Born law Diagonal anchoring and readout instances of *(R)^{⋆}*.

Partial-register Born law Whole-register law and *(C)^{⋆}*.

Projective residual Born law, *(C)^{⋆}*, continuation instances of *(R)^{⋆}*, fixed gates, and state sufficiency.

bottomrule

The anchoring proofs are alternatives. Nullity is not needed for locality. Continuity is used in anchoring, not as an update-continuity premise.

## Locality and uniform correlations

Throughout this and the next two sections, $S$ is unknown.

**Lemma.** Under *(NS)^{⋆}*, appending fresh zero-state factors outside $S$ does not change $P_{\psi}^{S}$. Applying a unitary outside $S$ does not change it either.

*Proof.* Compare a remote allocation-and-return program with a remote identity program, and schedule allocation before local readout. No-signaling equates the local laws. State sufficiency identifies the post-allocation law with the law at the enlarged global state.

For a remote gate, compare its gate-and-return program with the identity program and schedule the gate first.

**Lemma.** Under *(NS)^{⋆}*, global pure states with the same reduced density operator on $S$ have the same $S$ -outcome law.

*Proof.* Pad the smaller environment by zero-state qubits until environment dimensions agree. Remote invariance leaves the law unchanged; consistent relabeling identifies the environment name sets. If

$$\rho_S = \sum_j r_j |e_j\rangle\langle e_j|$$

on its support, the purifications have the forms

$$\sum_j \sqrt{r_j}|e_j\rangle|f_j\rangle, \qquad \sum_j \sqrt{r_j}|e_j\rangle|g_j\rangle,$$

with orthonormal environment families. The isometry $f_j \mapsto g_j$ extends to an environment unitary. Apply remote invariance again.

Hence there is a well-defined function

$$F_S(\rho_S) = \mathrm{P}^{S}_{\psi}, \qquad \rho_S = \mathrm{tr}_{L\setminus S}|\psi\rangle\langle\psi|.$$

This partial trace is an algebraic construction from the stipulated vector, not a measurement-probability postulate.

**Lemma.** Under *(NS)^{⋆}*, *(K)^{⋆}*, $F_S$ is continuous on the density operators on $\mathscr{H}_S$.

*Proof.* Use the fixed-dimensional purification

$$|\psi_\rho\rangle = (\sqrt{\rho} \otimes I)\sum_j |j\rangle|j\rangle.$$

Its norm is one and its reduction is $\rho$. Matrix square root is continuous in finite dimension, so this is a continuous choice of vector purification. Apply *(K)^{⋆}* and purification locality.

**Theorem.** Assume *(C)^{⋆}*, *(NS)^{⋆}*, *(N)^{⋆}*. If

$$\psi = \frac{1}{\sqrt{d}} \sum_{k \in K} e^{i\theta_k} |k\rangle_S |\eta_k\rangle_B, \qquad d = |K|,$$

where the computational strings $k$ are distinct and the computational strings $\eta_k$ are distinct, then $P_{\psi}^S$ is uniform on $K$.

*Proof.* A diagonal environment unitary removes the phases without changing the $S$-law. Write

$$p = \mathrm{P}_\psi^S, \qquad q = \mathrm{P}_\psi^B, \qquad \nu = \mathrm{P}_\psi^{S \cup B}.$$

Nullity restricts $\nu$ to $(k, \eta_k)$. Composability gives

$$p(k) = \nu(k, \eta_k) = q(\eta_k), \qquad \sum_{k \in K} p(k) = 1.$$

For a permutation $\pi$ of $K$, extend it to a computational-basis permutation unitary $\Pi_{\pi}$ on $S$. Then

$$\Pi_\pi \psi = \frac{1}{\sqrt{d}} \sum_{m \in K} |m\rangle |\eta_{\pi^{-1}(m)}\rangle.$$

The same vector is $W\psi$, where $W$ is an environment unitary sending $\eta_k$ to $\eta_{\pi^{-1}(k)}$. Its $S$-law remains $p$, by invariance under $W$. Its $B$-law remains $q$, by invariance under $\Pi_{\pi}$.

For the new pairing, nullity and composability imply

$$p(m) = q(\eta_{\pi^{-1}(m)}) = p(\pi^{-1}(m)).$$

Every permutation preserves $p$, so $p(k) = 1/d$.

Probability covariance under arbitrary basis permutations was not assumed. The required equality was derived by comparing two coherent implementations of the same transformed vector.

## Diagonal anchoring: independent finite proofs

Diagonal anchoring is the conclusion

$$\rho_S \text{ diagonal} \quad \Rightarrow \quad F_S(\rho_S)(\kappa) = \langle \kappa | \rho_S | \kappa \rangle.$$

It is not a separate requirement.

### A three-qubit illustration

**Proposition.** Assume *(C)^{⋆}*, *(N)^{⋆}* and (S5). For

$$\Omega = \frac{|011\rangle + |101\rangle + |110\rangle}{\sqrt{3}},$$

the first-qubit law is *(1/3, 2/3)*, and the first-qubit reduction is *diag (1/3, 2/3)*.

*Proof.* Nullity restricts the full law to the displayed strings. Tensor-factor permutations preserve $\Omega$ and act transitively on its support. Equivariance forces equal weights, hence *1/3* each. Composability gives the first-coordinate marginal. The accompanying last-two-qubit strings are distinct, so the first-qubit reduction has no off-diagonal entry and has the stated diagonal.

This example illustrates the mechanism; it does not by itself establish arbitrary diagonal anchoring.

### Proof A: multiset symmetry

**Theorem.** Assume *(C)^{⋆}*, *(NS)^{⋆}*, *(N)^{⋆}*, *(K)^{⋆}* and (S5). Then diagonal anchoring holds.

*Proof.* Let $S$ have $n$ qubits. First take a rational diagonal state

$$\rho_S = \sum_{k \in K} \frac{m_k}{M} |k\rangle\langle k|, \qquad m_k \geq 1, \qquad \sum_k m_k = M.$$

Zero-weight labels are omitted.

Take $M$ blocks of $n$ qubits. Let $\mathscr{A}$ consist of sequences $(k_1, \ldots, k_M)$ containing exactly $m_k$ copies of each $k$, and put

$$\Omega_{\mathbf{m}} = \frac{1}{\sqrt{|\mathscr{A}|}} \sum_{a \in \mathscr{A}} |k_1\rangle \cdots |k_M\rangle.$$

Block permutations are tensor-factor permutations. They preserve this vector and act transitively on $\mathscr{A}$. Nullity and equivariance force a uniform full law.

Composability gives the first-block marginal

$$\Pr(k_1 = k) = \frac{|\{a \in \mathscr{A} : a_1 = k\}|}{|\mathscr{A}|} = \frac{m_k}{M},$$

since

$$|\mathscr{A}| = \frac{M!}{\prod_\ell m_\ell!}, \qquad |\{a : a_1 = k\}| = \frac{(M-1)!}{(m_k - 1)! \prod_{\ell \neq k} m_\ell!}.$$

The first-block reduction is exactly $\rho_S$. Completions with different first labels have different remaining multisets, hence disjoint computational supports; all off-diagonal first-block entries vanish. The diagonal entries are the same counting ratios.

Locality transfers the law to every purification of $\rho_S$. Rational diagonal states are dense, so reduced-state continuity proves the result for all diagonal states. The empty register has a unique outcome.

This witness uses $nM$ qubits. Its resource size is finite but not uniformly bounded in the rational denominator.

**Why naive fine-graining does not suffice**

For disjoint label sets $R_k$, $|R_k| = m_k$, the vector

$$\Phi_0 = \frac{1}{\sqrt{M}} \sum_k \sum_{r \in R_k} |k\rangle_S |r\rangle_{E'}$$

has the desired reduction on $S$. However,

$$\rho_{E'} = \frac{1}{M} \sum_k \sum_{r, r' \in R_k} \mathrm{ketbra}\, r r'.$$

Coherences remain within each block. Distinct refined labels $r$ have repeated companions $k(r)$; Theorem therefore does not apply to the $E'$-readout.

A further computational label distinguishes those companions. This chooses another purification of the same $\rho_S$; it does not claim that a remote unitary decoheres a fixed reduced state.

**Proof B: decohered fine-graining**

**Theorem.** Assume $(C)^\star$, $(NS)^\star$, $(N)^\star$, $(K)^\star$. Then diagonal anchoring holds.

*Proof.* Use the rational weights above and set

$$a = \lceil \log_2 M \rceil, \qquad b = \left\lceil \log_2 \max_k m_k \right\rceil.$$

Choose disjoint $R_k \subseteq \{0,1\}^a$, $|R_k| = m_k$, and injections

$$e_k : R_k \to \{0,1\}^b.$$

Define

$$\Phi = \frac{1}{\sqrt{M}} \sum_k \sum_{r \in R_k} |k\rangle_S |r\rangle_{E'} |e_k(r)\rangle_{E''}.$$

Its reduction on $S$ is $\rho_S$.

With $E'$ written first, each $r$ has companion $(k(r), e_{k(r)}(r))$. These companions are distinct: different blocks have different $k$, and each $e_k$ is injective. Uniform correlations therefore give

$$\mathrm{P}_\Phi^{E'}(r) = 1/M$$

on the used labels.

Let *v = P_{Φ}^{S ∪ E^{'}}*. Nullity restricts its support to *(k(r), r)*. Composability makes its *E^{'}*-marginal uniform. Each *r* has only one allowed companion, so

$$v(k(r), r) = 1/M.$$

The *S*-marginal is *m_k/M*. Locality transfers this law to every purification of *ρ_S*, and continuity supplies arbitrary diagonal weights.

The witness uses

$$n + \lceil \log_2 M \rceil + \left\lceil \log_2 \max_k m_k \right\rceil$$

qubits. For weights *(1/3, 2/3)*, one choice is

$$\frac{1}{\sqrt{3}}(|0\rangle|00\rangle|0\rangle + |1\rangle|01\rangle|0\rangle + |1\rangle|10\rangle|1\rangle).$$

The three-qubit symmetry witness is smaller for that particular law.

The numerical fine-graining argument does not use (S5) as a probability-symmetry premise. The structural class still contains (S5); no independence theorem for the complete assumption package is claimed.

## Born laws and destructive projective residuals

### Whole-register records precede marginalization

**Theorem.** Assume diagonal anchoring, *(C)^{⋆}*, and the readout instances of *(R)^{⋆}*. Then, for every finite configuration and $S \subseteq L$,

$$\mathrm{P}^S_\psi(\kappa) = \left\| M^S_\kappa \psi \right\|^2.$$

*Proof.* First let $S = L$. The source

$$t_{L,*} = \mathrm{let}\, x = \mathrm{Enter}(\vec{q}_L)\ \mathrm{in}\ \langle *, x \rangle$$

is well typed because no complementary quantum names remain.

Write $\psi = \sum_j a_j |j\rangle_L$. Before record readout, the deferred program has vector

$$\Psi' = \sum_j a_j |j\rangle_L |j\rangle_R.$$

Its record reduction is

$$\rho_R = \sum_j |a_j|^2 \mathrm{ketbra}\, jj.$$

Diagonal anchoring gives record law $|a_j|^2$. The readout record equation identifies this with $P_{\psi}^{L}(j)$.

For a proper subset, composability gives

$$\begin{aligned} \mathrm{P}^S_\psi(\kappa) &= \sum_{\lambda \in \{0,1\}^{L \setminus S}} \mathrm{P}^L_\psi(\kappa, \lambda) \\ &= \sum_\lambda |a_{\kappa\lambda}|^2 = \left\| M^S_\kappa \psi \right\|^2. \end{aligned}$$

No partial-register record test with omitted spectators is used.

### Residual identification by future tests

**Theorem.** Assume the Born outcome law, *(C)^{⋆}*, and the continuation instances of *(R)^{⋆}*. If

$$\psi = \sum_\kappa |\kappa\rangle_S |\xi_\kappa\rangle_B, \qquad p_\kappa = \|\xi_\kappa\|^2 > 0,$$

then

$$\mathrm{U}^S_\psi(\kappa) \mathrm{doteq} \frac{\xi_\kappa}{\|\xi_\kappa\|}.$$

*Proof.* Fix a positive branch *κ*. Choose unit representatives

$$u_\kappa = \mathrm{U}_\psi^S(\kappa), \qquad \varphi_\kappa = \xi_\kappa/\|\xi_\kappa\|.$$

If $B = \varnothing$, its Hilbert space is one dimensional and the claim is immediate.

For a unitary $V$ on $B$, use

$$t_V = \mathrm{let}\, x = \mathrm{Enter}(\vec{q}_S)\ \mathrm{in?}\langle \mathrm{Enter}(V(\vec{q}_B)), x\rangle.$$

By the unknown strategy's transition rule and state sufficiency, the joint source weight for $(\lambda, \kappa)$ is

$$p_\kappa\, \mathrm{P}^B_{Vu_\kappa}(\lambda).$$

This does not use a Born update.

The deferred vector after copying and applying $V$, before either readout, is

$$\Psi'_V = \sum_j |j\rangle_S (V\xi_j)_B |j\rangle_R.$$

The target reads $B$, then $R$, and returns $S$. Composability equates its joint law to joint $B \cup R$ readout. The established Born outcome law gives joint weight

$$|\langle\lambda|V|\xi_\kappa\rangle|^2.$$

Records therefore imply

$$\mathrm{P}^B_{Vu_\kappa}(\lambda) = |\langle\lambda|V|\varphi_\kappa\rangle|^2.$$

Using the established outcome law on the left,

$$|\langle\lambda|V|u_\kappa\rangle|^2 = |\langle\lambda|V|\varphi_\kappa\rangle|^2$$

for every $V$, $\lambda$.

Choose $V\varphi_{\kappa} = |0^{|B|}\rangle$. Then

$$\left|\langle 0^{|B|} \mathrm{rvert V}\, |u_\kappa\rangle\right|^2 = 1,$$

so the unit rays agree.

The test unitary may be selected separately for each fixed branch. Each individual experiment still has a continuation independent of its first outcome. Equality of the *joint* reported laws permits conditioning on that outcome after the experiment. No single unitary must map all conditional vectors to one vector.

*Proof.* Inhabitation is Theorem . For an arbitrary model of the five requirements, Theorem gives diagonal anchoring, Theorem gives Born probabilities, and Theorem gives every positive-branch residual ray. These are precisely the data of the strategy.

**Corollary.** Every model of $T$ has the liberal observational confluence of *EC_Born*.

*Proof.* Its transitions have been identified with the Born transitions. Apply Theorem .

### Dense coherent gate languages

**Corollary.** The characterization remains valid when exact CNOT is available and finite available gate circuits are dense, up to phase, in the unitary group on every finite register, with all other assumptions retained.

*Proof.* Remote invariance first holds for available circuits. At fixed dimension, approximate an arbitrary remote unitary and use $(K)^{\star}$ to extend the invariance to its limit. This supplies the transformations required for locality and uniform correlations. Witness rays are admissible inputs regardless of exact preparability.

Record terms use exact CNOT. In the update proof, take available $V_j$ converging to a unitary mapping $\varphi_{\kappa}$ to $|0\rangle$. The equal squared-amplitude identities pass to the limit. No continuity of the unknown update map is needed.

## Stability and represented measurements

### Resource-budgeted record stability

For finite laws and unit rays, define

$$d_{\mathrm{TV}}(P,Q)=\frac{1}{2}\sum_x |P(x)-Q(x)|, \qquad D(u,v)=\sqrt{1-|\langle u\rangle v|^2}.$$

The latter is the trace distance between the corresponding pure-state density operators.

The next theorem concerns the record stage only. It does not derive approximate anchoring from approximate versions of the first four requirements.

**Theorem.** Fix a total live-qubit budget $N$ and $\varepsilon \geq 0$. Suppose a strategy in the structural class satisfies exact $(C)^{\star}$ and:

• For every configuration with at most $N$ live qubits and diagonal measured reduction,

$$d_{\mathrm{TV}}(\mathrm{P}^S_\psi, \operatorname{diag}\rho_S)\leq \varepsilon.$$

• Every record source and target whose executions remain within $N$ live qubits obey

$$d_{\mathrm{TV}}(\mathrm{Obs_S}\,[\psi, t_{S,u}], \mathrm{Obs_S}\,[\psi, \mathrm{defer}_S\,(t_{S,u})])\leq \varepsilon.$$

Then:

• If $\psi$ has $n$ live qubits and $2n \leq N$, then

$$d_{\mathrm{TV}}(\mathrm{P}^S_\psi, \mathrm{Born}^S_\psi)\leq 2\varepsilon$$

for every $S$, where $\mathrm{Born}_{\psi}^{S}(\kappa) = \|M_{\kappa}^{S}\psi\|^2$.

• If $|S| = k$ and $2(n + k) \leq N$, the joint law of

$$\mathrm{let}\, x = \mathrm{Enter}(\vec{q}_S)\ \mathrm{in?}\langle \mathrm{Enter}(V(\vec{q}_{L\setminus S})), x\rangle$$

is within $3\varepsilon$ of its Born joint law for every $V$.

• Under the budget in (b), if

$$b=\left\|M^S_\kappa\psi\right\|^2\geq \gamma>0, \qquad \varepsilon\leq \gamma/8,$$

the branch has positive strategy probability, and

$$D\left(\mathrm{U}^S_\psi(\kappa), \frac{M^S_\kappa\psi}{\sqrt{b}}\right)\leq \min\left\{1, \sqrt{\frac{6\varepsilon}{\gamma}}\right\}.$$

*Proof.* For (a), the whole-register record experiment adds $n$ fresh qubits. Approximate records cost at most $\varepsilon$; approximate anchoring of the record reduction costs at most another $\varepsilon$. Exact composability and contraction under marginalization give every measured subset.

For (b), comparison with the deferred experiment costs $\varepsilon$. Before deferred readout, the vector is

$$\Psi'_V=\sum_j |j\rangle_S (V\xi_j)_B |j\rangle_R$$

on $n + k$ qubits. Exact composability replaces the successive $B, R$ readouts by a joint readout. Part (a), applied to this intermediate vector, costs $2\varepsilon$ and requires $2(n + k) \leq N$. The Born weight is $|\langle\lambda|V|\xi_{\kappa}\rangle|^2$. The triangle inequality gives $3\varepsilon$.

For (c), write

$$p=\mathrm{P}^S_\psi(\kappa), \quad u=\mathrm{U}^S_\psi(\kappa), \quad \varphi=M^S_\kappa\psi/\sqrt{b}.$$

Part (a) yields $|p - b| \leq 2\varepsilon$, hence $p \geq 3\gamma/4 > 0$. If the complement is empty, its residual ray is unique. Otherwise choose $V\varphi = |0\rangle$. The Born joint probability of $\kappa$ followed by $\lambda \neq 0$ is zero, so (b) gives

$$p\,\mathrm{P}^B_{Vu}(\lambda\neq 0)\leq 3\varepsilon.$$

Apply (a) to $Vu$; its required budget is covered by the assumed larger budget. Then

$$\begin{aligned} D(u,\varphi)^2 &= 1-|\langle 0|V|u\rangle|^2 \\ &\leq \frac{3\varepsilon}{p}+2\varepsilon \\ &\leq \frac{4\varepsilon}{\gamma}+\frac{2\varepsilon}{\gamma}=\frac{6\varepsilon}{\gamma}, \end{aligned}$$

using $\gamma \le 1$. Finally $D \le 1$.

A positive lower branch-weight bound is necessary for this form of conditional estimate: a small unconditional discrepancy may be concentrated on a rare branch. No general repeated-deferral error theorem is inferred.

**Mixed states and finite adaptivity**

**Corollary.** In a model of $T$, evaluation through a purification of a laboratory density operator $\rho$ gives

$$\Pr(\kappa) = \operatorname{tr}(P^S_\kappa \rho), \qquad \rho'_\kappa = \frac{\operatorname{tr}_S(P^S_\kappa \rho P^S_\kappa)}{\operatorname{tr}(P^S_\kappa \rho)}$$

on positive branches. Both expressions are independent of the purification.

*Proof.* Apply the probability and update theorems to a purification. Trace its inaccessible purifying factors from the conditioned residual. The formulas depend only on $\rho$. Purifying names can be returned as inert spectators in the total pure configuration; no term-level discard is needed.

**Corollary.** Every finite program with a first-order interface in a model of $T$ has its Born instrument. Its terminal classical observations are independent of the liberal scheduler.

*Proof.* Characterization identifies each reached readout's probabilities and positive-branch residuals. Induction on the finite remaining evaluation tree identifies all path maps and reported-output instruments. Scheduler independence follows from Corollary .

**Rotated readout and POVMs**

For $|u_{\kappa}\rangle = V|\kappa\rangle$, define the macro

$$\mathrm{Enter}_V(\vec{q}_S) = \mathrm{Enter}(V^\dagger(\vec{q}_S)).$$

Its law is $\langle u_{\kappa}|\rho_S|u_{\kappa}\rangle$, and its residual is proportional to $(\langle u_{\kappa}|_S \otimes I)\psi$.

For a finite POVM $(E_x)_x$, the map

$$V_E\varphi = \sum_x \sqrt{E_x}\varphi \otimes |x\rangle$$

is an isometry because $\sum_x E_x = I$. After finite qubit encoding and zero-state padding, it extends to a unitary. Reading the outcome register and returning the quantum residual implements

$$\Pr(x) = \operatorname{tr}(\rho E_x).$$

A finite classical decoder handles non-power-of-two label sets. Unused padded labels have zero weight on the encoded input subspace. This is one implementation; the POVM alone does not specify all possible conditional maps.

**Corollary.** Let

$$\Phi_x(\rho) = \sum_j M_{xj}\rho M_{xj}^\dagger, \qquad \sum_{x,j} M_{xj}^\dagger M_{xj} = I$$

be a finite-dimensional instrument with fixed input and quantum output spaces. After finite qubit encodings, an Enter program realizes these maps on its declared classical and quantum outputs. With an untouched reference $B$, its branch map is $\Phi_x \otimes id_B$.

*Proof.* Let $J_{in}$ and $J_{out}$ be isometric qubit encodings of the input and output spaces. Choose a qubit register large enough to accommodate the encoded output, a computational register for $(x, j)$, and zero padding. On the encoded zero-ancilla input subspace prescribe

$$J_{\mathrm{in}}\varphi \otimes |0\rangle \mathrm{longmapsto} \sum_{x,j} J_{\mathrm{out}} M_{xj}\varphi \otimes |x,j\rangle \otimes |0\rangle_{\mathrm{pad}}.$$

Completeness makes this prescription isometric. Enlarge the zero-ancilla input register if necessary so that input and target are subspaces of the same finite qubit space. Complete orthonormal bases to extend the prescription to a unitary.

Implement the unitary, destructively read the index and padding registers, decode $x$, delete unreported classical indices by *dropBit*, and return the encoded quantum output. On the encoded input, the resolved path maps are the

encoded *M_xj*, with zero padding fixed. Summing the unreported *j* gives *Φ_x*. Every operation acts trivially on *B*, proving the reference claim.

This theorem represents a specified instrument by a program after the primitive is fixed. It does not characterize an independently declared instrument primitive or derive a physical apparatus implementing the program.

## After Enter: records, instruments, decoders, and cutoffs

This section is an interface dictionary. It adds no premise to Theorem . Its finite statements are proved below; companion citations identify related source records rather than transfer unproved apparatus or field results into the characterization.

### D1: three distinct measurement interfaces

After characterization, write *L = Ssqcup B*. A classical register is represented below by a diagonal matrix algebra. It is not an additional live quantum carrier in the term language merely because Hilbert-space notation is used for its labels.

The three interfaces distinguished in Cheung's En03-A [CheungEn03A] are:

**Destructive Enter.** Its complete classical/quantum channel is

$$\mathscr{M}_S(\rho) = \sum_z \text{ketbra}\, zz_R \otimes M_z^S \rho (M_z^S)^\dagger.$$

The output residual is *B*; the factor *S* is absent.

**Coherent recording.** The copying isometry is

$$A_S = \sum_z |z\rangle_R \otimes P_z^S, \qquad A_S^\dagger A_S = I,$$

and the coherent channel is

$$\mathscr{C}_S(\rho) = A_S \rho A_S^\dagger.$$

Both *S* and the quantum record carrier *R* remain. Off-diagonal coherences in *R* are retained in the joint state. This is not yet a classical readout.

**Represented nondestructive projective instrument.** Copying and then destructively reading the record carrier implements

$$\mathscr{I}_S^{\text{nd}}(\rho) = \sum_z \text{ketbra}\, zz_R \otimes P_z^S \rho P_z^S.$$

Its classical record is *R*, and its quantum residual interface is $S \otimes B$. It is represented by allocation, CNOT, and Enter; it is not the destructive primitive.

These formulas follow from the characterized semantics. They do not constitute another derivation or axiom of the Born rule.

**Definition.** A record is a classical value already produced by Enter or obtained by reading a coherent record carrier, together with a declared continuation on the surviving quantum names. Retained names remain separately owned. Equality of classical records does not identify all coherent implementations or their residual instruments.

### D2: record pullback versus instrument compatibility

**Proposition.** Let *T* be a finite-dimensional CPTP map from a fine input space to a coarse space, let *(E_w)_w* be a coarse POVM, and let *(Π_w)_w* be a fine POVM with the same labels. The record laws

$$\text{tr}\,(T(\rho)E_w), \qquad \text{tr}\,(\rho \Pi_w)$$

agree for every input density operator and every *w* if and only if

$$T^*(E_w) = \Pi_w \qquad \text{for every} w.$$

Here *T^{*}* is the trace-dual map.

*Proof.* The difference is

$$\operatorname{tr}(\rho[T^{*}(E_w) - \Pi_w]).$$

The bracket is Hermitian. Vanishing expectation on all density operators, in particular its eigenvector states, is equivalent to the bracket being zero. The reverse implication is immediate.

This is the finite pullback criterion recorded in En03-A [CheungEn03A]. It determines effects, not complete selective maps.

For instruments *(Φ_w)_w* and *(Ψ_w)_w* with identified residual output spaces, full implementation compatibility is instead

$$\Psi_w \circ T = \Phi_w \qquad \text{for every} w.$$

Equality on all matrix units certifies this map identity and therefore also equality with every untouched reference. Its trace consequence is record compatibility, but the converse fails.

**Example.** For a qubit input and $z \in \{0, 1\}$, define

$$\Phi_z(\rho) = \langle z|\rho|z\rangle \operatorname{ketbra} 00, \qquad \Psi_z(\rho) = \langle z|\rho|z\rangle \operatorname{ketbra} 11.$$

Both are complete instruments with effects *ketbra zz*. Thus they have identical Born record laws for every input. Their conditional quantum outputs differ on every positive branch.

Both instruments are represented by Enter followed by allocation and, for $\Psi$, a fixed $X$ gate on the fresh output. They do not define two different Enter strategies. A later computational readout of the returned output distinguishes them perfectly.

Thus record compatibility is a pullback, whereas residual compatibility requires a separate instrument certificate. The example supplies a self-contained instance of the record/residual distinction associated with En03-A [CheungEn03A].

**Definition.** A compiled-instrument interface consists of:

• a finite Enter program and declared classical and quantum outputs;

• a dependency graph recording classical inputs to operations and order constraints imposed by shared quantum resources;

• complete selective maps at its boundary;

• a certificate of the exact composition, exchange, or implementation identities claimed. Correctness concerns the complete selective maps, not only outcome effects. Existence of a physical implementation certificate is additional data; no general external-apparatus compiler theorem is asserted.

### Complete process data: the direction of implication

**Proposition.** Consider two finite adaptive models on identified accessible spaces. Suppose their initial accessible states agree and, after every permitted setting and history, their interval channels and complete selective instruments agree. Then all permitted adaptive record laws agree, including with an untouched reference. For an equal-prior binary randomized test, the type-I and type-II errors satisfy

$$\alpha + \beta = 1.$$

For arbitrary record laws *P_0, P_1*,

$$\alpha + \beta \geq 1 - d_{\mathrm{TV}}(P_0, P_1).$$

*Proof.* Induction through the finite experiment tree gives equality of every unnormalized accessible conditional state. Taking its trace gives equality of history weights and hence record laws. Tensoring the assumed map identities with the reference identity gives the same induction with a reference.

If $\varphi \in [0, 1]$ is the probability of deciding for model *1*, then

$$\alpha + \beta = \mathbb{E}_0\varphi + \mathbb{E}_1(1 - \varphi) = 1 + \mathbb{E}_0\varphi - \mathbb{E}_1\varphi.$$

The difference of expectations has magnitude at most total variation.

This is the implication supplied by Psy02-B's thm:process [CheungPsy02B]. It does *not* state the converse. In particular, that theorem alone is not a residual-instrument nonidentifiability theorem.

Whether equal records identify a residual map depends on the permitted future experiments and accessible output algebra. If arbitrary tomographic tests of an identified accessible residual are allowed, differences such as Example are detectable. If the comparison boundary excludes those outputs or tests, record equality need not identify them.

### D3: finite experiments, resource ledgers, and cutoffs

**Definition.** A finite adaptive experiment is a finite tree of Enter-typed operations with a finite live-resource bound. Controls use only records already produced on the current branch. Its resource ledger states which quantum names are charged, when inputs are delivered, and how surviving and returned outputs are counted.

The stability budget *N* in Theorem counts the *total global* live qubits of its configurations. A scheduled decoder's *laboratory peak* may use a different declared accounting convention, including separate external inputs and references. These quantities are not interchangeable.

Under already specified Born semantics, a finite resource-certified decoder is a program, not a new primitive. Its complete branch sum certifies a channel on arbitrary reference-entangled inputs; its live-name ledger certifies the declared peak. This is the soundness interface in En02-B [CheungEn02B].

The same channel can have different resource costs under different legal schedules. In En02-B's declared perfect-code tree and serial module model, a retained-coherent convention requires peak *3^h*, whereas its adaptive module construction has peak *h + 1* [CheungEn02B]. These are model-specific bounds, not lower bounds for every Enter implementation or an alteration of the measurement strategy.

**Definition.** An apparatus-cutoff comparison specifies:

• a full model and a retained model;

• state encodings and identified output interfaces;

• paired operations and a declared input class;

• the experiment and resource budgets;

• a comparison certificate, for example

$$d_{\mathrm{TV}}(\mu_{\mathrm{full}}, \mu_{\mathrm{retained}}) \le \delta.$$

A spectral cutoff must have a finite-rank retained space with a finite qubit encoding before it is an Enter configuration. Spectral projection alone does not guarantee finite rank. A certificate for one prepared input law is not automatically a uniform channel or reference-preserving certificate.

No cutoff estimate is a premise of the characterization theorem.

### D4: the boundary of primitive uniqueness

Primitive uniqueness concerns the state-sufficient pair *(P, U)*. A represented instrument, decoder, or cutoff comparison is evaluated after that strategy is fixed; it is not a replacement strategy.

A rule consulting an unrepresented apparatus identifier, cutoff parameter, or execution history is outside the structural class. If such a parameter is represented by ordinary live resources, the theorem applies to the resulting full configuration, not to an undeclared parameter-dependent primitive.

**Proposition.** Suppose the only accessible output of an input system is a finite classical measurement record. Any channel obtained by decoding that record into a quantum output has the form

$$\Lambda(\rho) = \sum_z \mathrm{tr}\,(E_z \rho)\sigma_z$$

and is entanglement breaking. It cannot equal the identity channel on a code of dimension at least two.

*Proof.* With a reference, the output is a sum of positive reference conditional operators tensored with *σ_z*, hence is separable. The identity channel preserves a maximally entangled state on a code of dimension at least two, whose output is not separable.

Hawk01-B gives a model-specific number-record recovery obstruction of this kind [CheungHawk01B]. A number-diagonal record therefore is not, by itself, a coherent recovery or residual-leakage certificate.

Likewise, agreement of thermal marginals or spectra does not specify a full transfer channel. The finite conserving completions of Hawk01-A and the spectrum-matching separation of Hawk02-A are documentary examples

[CheungHawk01A] [CheungHawk02A]. Their physical hypotheses are not imported into this paper.

par **External physical and analytic boundaries** par The supplied Em03-C source record distinguishes exterior finite-mode tests from interior partner constructions and explicitly does not resolve horizon smoothness or the firewall question [CheungEm03C]. The finite key-transfer model Em03-E supplies a specified channel and recovery analysis, not a microscopic horizon derivation [CheungEm03E].

The supplied XIV-I record constructs its named internal amplitude at I2 but leaves the joint characteristic operator map unresolved at I3; its release does not admit the separate quantum coefficient [CheungXIVI]. These are documentary boundaries, not premises here. Nothing in the present calculus discharges XV-B, XVI-N, or O.1. No theorem is attributed to an unavailable companion excerpt, and an unconstructed object is not thereby proved nonexistent. par

## Context refinement and the Frauchiger–Renner example

This section gives a finite observational-context calculation, related to the context discussion of Cheung [EnterCalculus] and the scenario of Frauchiger and Renner [FrauchigerRenner]. It is not used in primitive characterization.

### Sharp contexts and transport

A sharp context is a finite PVM. Context $D$ refines $C$ through a label map $f$ if

$$\Pi_{C,a} = \sum_{d\,:\,f(d)\,=\,a} \Pi_{D,d}.$$

**Lemma.** Two finite PVMs have a common sharp refinement if and only if every projector of one commutes with every projector of the other.

*Proof.* A common refinement expresses both families as sums of one orthogonal projector family, so the sums commute. Conversely, commuting projectors give the joint family

$$\Pi_{a,b} = \Pi_{C,a}\Pi_{D,b}.$$

These are orthogonal projections summing to the identity and refine both PVMs.

For events $E$, $F$ in one context, write

$$\Psi \mathrm{vdash}_C\, E \Rightarrow F \quad \text{if} \quad \Pr_C(E \setminus F) = 0.$$

Chaining is sound within one context, because

$$E \setminus G \subseteq (E \setminus F) \cup (F \setminus G).$$

Transport to a refining context uses the specified inverse-image map on labels, with the same event projectors. Shared verbal labels alone do not provide such a refinement.

### The finite two-laboratory calculation

Take

$$\Psi = \frac{1}{\sqrt{3}}\big(|\bar{h}\rangle|\mathrm{Downarrow}\rangle + |\bar{t}\rangle|\mathrm{Downarrow}\rangle + |\bar{t}\rangle|\mathrm{Uparrow}\rangle\big),$$

and define

$$|\overline{\mathrm{ok}}\rangle = \frac{|\bar{h}\rangle - |\bar{t}\rangle}{\sqrt{2}}, \qquad |\mathrm{ok}\rangle = \frac{|\mathrm{Downarrow}\rangle - |\mathrm{Uparrow}\rangle}{\sqrt{2}}.$$

The fail vectors use plus signs.

Let

$$\begin{aligned} llC_{\mathrm{rec}}: &\quad \{\bar{h},\bar{t}\} \otimes \{\mathrm{Downarrow},\mathrm{Uparrow}\},\\ C_{\overline{W}}: &\quad \{\overline{\mathrm{ok}},\overline{\mathrm{fail}}\} \otimes \{\mathrm{Downarrow},\mathrm{Uparrow}\},\\ C_W: &\quad \{\bar{h},\bar{t}\} \otimes \{\mathrm{ok},\mathrm{fail}\}. \end{aligned}$$

The corresponding zero-probability events give

$$\begin{aligned} llC_{\mathrm{rec}}: &\quad \mathrm{Uparrow} \Rightarrow \bar{t}, \\ C_{\overline{w}}: &\quad \overline{\mathrm{ok}} \Rightarrow \mathrm{Uparrow}, \\ C_{w}: &\quad \bar{t} \Rightarrow \mathrm{fail}. \end{aligned}$$

Indeed,

$$\langle \bar{h}|\langle \mathrm{Uparrow}|\Psi = 0, \qquad \langle \overline{\mathrm{ok}}|\langle \mathrm{Downarrow}|\Psi = 0, \qquad \langle \bar{t}|\langle \mathrm{ok}|\Psi = 0.$$

However,

$$[|\bar{t}\rangle\langle\bar{t}|, |\overline{\mathrm{ok}}\rangle\langle\overline{\mathrm{ok}}|] \neq 0.$$

The necessary contexts therefore have no common sharp refinement. In the outer context measuring both interference bases,

$$\Pr(\overline{\mathrm{ok}}, \mathrm{ok}) = \left|\frac{1}{2\sqrt{3}}\right|^2 = \frac{1}{12}.$$

Transporting the three implications into one outer classical chain is not justified by the stated refinement discipline.

This is an observational-context obstruction. It is not a theorem that linear ownership alone rejects all cross-observer inference. Sequential operations on the same laboratory can be well typed and can disturb later tests.

No interior observable algebra or gravitational dynamics is specified here. The calculation consequently establishes no claim about smooth horizons or firewalls. This boundary agrees with the explicit finite-mode non-claims of Em03-C [CheungEm03C]; the physical firewall question has a different hypothesis package [AMPS].

An internal syntax for observer judgments, certificates, and allowed transports remains P3 below.

## Limits and successor problems

The closed result is Theorem . The structural class is declared, not reconstructed from its five requirements. The first four requirements give diagonal anchoring at arbitrary finite sizes. Records connect coherent whole-register readout to those diagonal laws; composability gives partial readout; continuation tests give positive-branch residual rays.

The proof order does not establish that records are necessary. It also does not assert that the first four requirements already determine coherent probabilities or updates.

par **Successor problems** par **P1. Necessity of the record equation.** Does a strategy in Assumption satisfy *(C)^{⋆}, (NS)^{⋆}, (N)^{⋆}, (K)^{⋆}* but fail *(R)^{⋆}*? Separate: (i) possible freedom in coherent outcome laws; and (ii) possible freedom in conditional updates once the complete Born outcome law is fixed. No equivalence between these subquestions is presumed.

**P2. A weaker route to record-free uniqueness.** Without assuming the record equation, can a strictly weaker, independently motivated operational condition identify coherent laws and updates from diagonal anchoring? The fixed language already defines the record term using allocation and CNOT. Removing that definable construction would change the structural problem. ``Record-free'' here means omitting its observational equation.

**P3. Internal context-refinement discipline.** Add syntax for observer judgments, refinement certificates, and permitted transports. Under a specified encoding, prove soundness and underivability of the Frauchiger–Renner certainty chain. Section supplies the finite projector obstruction, not this internalization. par

En03-A's schedule-locality interface bounds influence for prescribed dynamics and schedules, but does not supply the observer-judgment syntax or underivability theorem required by P3 [CheungEn03A]. Its residual-instrument distinction concerns implementation compatibility after a primitive is fixed, not P1 or P2. Similarly, the static family in Appendix fails operational nullity and is not a record-free counterstrategy.

The stability result assumes exact composability and approximate anchoring; approximate anchoring is not derived from noisy versions of the first four requirements. The finite instrument representation is mathematical

implementation inside the fixed semantics, not a physical apparatus existence theorem.

**Conclusion.** The calculus separates linear ownership from the global quantum state. Its specified Born semantics is finite, safe, and observationally confluent. When only measurement varies, the stated operational theory is inhabited and uniquely identifies Born probabilities and destructive projective residuals within its declared class. Represented measurements and later implementation interfaces remain logically downstream of that result.

appendix

## The independent static marginal problem

This appendix studies a weaker, independent problem. It assumes diagonal anchoring but has no updates, sequential experiments, or record equation. It neither proves nor refutes P1.

### Static assumptions

For every finite labeled qubit set $S$, consider

$$F_S : \mathscr{D}(\mathscr{H}_S) \to \mathbb{R}^{\{0,1\}^S},$$

where $\mathscr{D}(\mathscr{H}_S)$ denotes density operators. Require:

**Diagonal anchoring.** If $\rho$ is diagonal, $F_S(\rho) = diag\ \rho$.

**Marginal compatibility.** For $T \subseteq S$,

$$\mathrm{marg}_T F_S(\rho) = F_T(\rho_T).$$

**Continuity.** Every $F_S$ is continuous.

**Positivity, when imposed.** Every coordinate is nonnegative. Set $F_\{\varnothing\}(1) = 1$. Compatibility then implies normalization. Without positivity, call the family signed.

Algebraic partial-trace compatibility does not describe how a first measurement prepares a state for a second measurement.

### Signed slack representation

For $T \subseteq S$, define the Walsh character

$$\chi_T(x) = \prod_{i \in T} (-1)^{x_i}$$

and normalized coefficient

$$\hat{f}(T) = 2^{-|S|} \sum_x f(x)\chi_T(x).$$

Character orthogonality gives

$$2^{-|S|} \sum_x \chi_T(x)\chi_U(x) = \begin{cases} 1, & T = U, \\ 0, & T \neq U, \end{cases}$$

and hence

$$f(x) = \sum_{T \subseteq S} \hat{f}(T)\chi_T(x).$$

**Theorem.** Signed, anchored, marginally compatible, continuous families are in bijection with continuous functionals $c_T$, one for every nonempty labeled set $T$, vanishing on diagonal states, through

$$\mathrm{boxed}\, F_S(\rho)_x = \langle x|\rho|x\rangle + \sum_{\varnothing \neq T \subseteq S} 2^{|T|-|S|} c_T(\rho_T)\chi_T(x).$$

*Proof.* Put $Q_S(\rho) = F_S(\rho) - diag\ \rho$. Normalization gives $Q_S(\rho)(\varnothing) = 0$. Compatibility yields, for $T \subseteq S$,

$$\begin{aligned} \widehat{Q_T(\rho_T)}(T) &= 2^{-|T|} \sum_{x_T} Q_T(\rho_T)_{x_T} \chi_T(x_T) \\ &= 2^{-|T|} \sum_{x_S} Q_S(\rho)_{x_S} \chi_T(x_S) \\ &= 2^{|S|-|T|} \widehat{Q_S(\rho)}(T). \end{aligned}$$

Define

$$c_T(\sigma) = \widehat{Q}_T(\sigma)(T).$$

Walsh inversion gives the displayed representation and uniqueness. Continuity is inherited from *Q_T*, and anchoring makes *c_T* vanish on diagonal states.

Conversely, choose such functionals and use the formula. Every nonempty character sums to zero, giving normalization. For a diagonal *ρ*, all *ρ_T* are diagonal, so all corrections vanish.

To marginalize from *S* to *U*, characters with an index outside *U* cancel. For $T \subseteq U$, summation contributes the factor *2^{|S| − |U|}*, converting *2^{|T| − |S|}* to *2^{|T| − |U|}*. This is exactly the formula for *F_U(ρ_U)*. Continuity follows from continuous partial trace and the finitely many terms at each fixed size.

Positivity imposes additional inequalities; it is not contained in this signed classification.

**An all-size positive non-Born family**

**Theorem.** There exist normalized, positive, continuous, marginally compatible, diagonally anchored, permutation-equivariant static families different from the Born diagonal assignment.

*Proof.* Fix $0 < \eta < 1$. For a one-qubit state $\tau$, define

$$\theta(\tau) = \eta \operatorname{Re} \tau_{01}, \qquad W(\tau) = e^{-i\theta(\tau)Y}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

For $\rho$ on $S$, set

$$W_S(\rho) = \operatorname{bigotimes}_{i \in S} W(\rho_i), \qquad F_S(\rho)_x = \langle x | W_S(\rho) \rho W_S(\rho)^\dagger | x \rangle.$$

Use the unique normalized assignment for the empty set.

For fixed $\rho$, this is the diagonal of a density operator, so it is positive and normalized. Partial trace, matrix exponential, and diagonal extraction are continuous. If $\rho$ is diagonal, each local angle is zero, so anchoring holds. Using one identical rule at all sites gives permutation equivariance.

Define

$$E_{i,x_i}(\rho_i) = W(\rho_i)^\dagger |x_i\rangle\langle x_i \operatorname{rvertW}(\rho_i).$$

Then

$$F_S(\rho)_x = \operatorname{tr}\left(\rho \operatorname{bigotimes}_{i \in S} E_{i,x_i}(\rho_i)\right).$$

Summing over $S \setminus T$ gives

$$\begin{aligned} \sum_{x_{S \setminus T}} F_S(\rho)_x &= \operatorname{tr}\left(\rho \left[\operatorname{bigotimes}_{i \in T} E_{i,x_i}(\rho_i) \otimes I_{S \setminus T}\right]\right) \\ &= \operatorname{tr}\left(\rho_T \operatorname{bigotimes}_{i \in T} E_{i,x_i}((\rho_T)_i)\right) \\ &= F_T(\rho_T)_{x_T}, \end{aligned}$$

using local completeness and *(ρ_T)_i = ρ_i*.

Finally, for *ρ = ketbra + +*, the angle is *η/2*. Since

$$e^{-i\theta Y} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

$$F_{\{1\}}(\operatorname{ketbra} + +)_0 = \frac{1 - \sin\eta}{2} \neq \frac{1}{2}.$$

The family is non-Born.

The state-dependent basis in this construction defines a mathematical probability function. It is not asserted to be a fixed primitive operation in the Enter language.

**Explicit failure of operational nullity**

**Proposition.** The family of Theorem violates operational nullity on a two-qubit pure input.

*Proof.* Let

$$\psi = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}.$$

Its *11*-amplitude vanishes. Both one-qubit reductions have off-diagonal entry *1/3*, so both rotations have angle $\theta = \eta/3$.

Write $s = \sin\theta$, $c = \cos\theta$. Both are positive because $0 < \theta < 1/3$. Directly,

$$\langle 11|(W \otimes W)|\psi\rangle = \frac{s^2 + 2sc}{\sqrt{3}} > 0.$$

Therefore

$$F_{\{1,2\}}(|\psi\rangle\langle\psi|)_{11} = \frac{(s^2 + 2sc)^2}{3} > 0,$$

although *M_11^{{1, 2}}ψ = 0*.

No strategy with this full-register law can satisfy *(N)^{⋆}*, irrespective of its proposed residual maps. Thus this positive family is not a record-free operational countermodel.

**Limited static rigidity**

**Lemma.** If every proper reduced state of $\rho$ on $S$ is diagonal, then

$$F_S(\rho)_x = \rho_{xx} + c_S(\rho)\chi_S(x).$$

For a positive family, a zero diagonal entry at even parity forces *c_S(ρ) ≥ 0*, and one at odd parity forces *c_S(ρ) ≤ 0*. If both occur, *F_S(ρ) = diag ρ*.

*Proof.* All proper-set coefficients vanish by anchoring. At a zero diagonal entry, positivity imposes the corresponding sign on the sole remaining correction.

For $n \geq 3$, the state

$$a|0\rangle^{\otimes n} + b|1\rangle^{\otimes n}, \qquad |a|^2 + |b|^2 = 1,$$

has diagonal proper reductions. Absent strings of weights one and two have opposite parities. Every positive compatible family therefore agrees with the Born diagonal on this input, even though non-Born freedom exists elsewhere.

**Proposition.** At a fixed finite size, an affine, positive, normalized assignment on density operators that agrees with the diagonal on diagonal states is the Born diagonal assignment.

*Proof.* Every affine coordinate on the trace-one Hermitian hyperplane has the form *tr (E_xρ)* for a Hermitian *E_x*; its affine constant can be absorbed into a multiple of the identity. Positivity on all states gives *E_x ≥ 0*. Normalization gives *∑_x E_x = I*.

Anchoring on basis states gives

$$\langle y|E_x|y\rangle = \delta_{xy}.$$

For a positive matrix,

$$|(E_x)_{yz}|^2 \leq (E_x)_{yy}(E_x)_{zz}.$$

All off-diagonal entries vanish, and the only nonzero diagonal entry is the *x*-entry, equal to one. Thus *E_x = ketbra xx*.

The positive family above is necessarily nonlinear. Static compatibility and continuity do not imply affinity. Adding affinity would be a new static hypothesis, not a premise silently used in the main proof.

**Logical separation.** The signed theorem classifies static corrections. The positive theorem shows non-Born static assignments at all sizes. The nullity proposition explains why this particular family cannot interpret the operational primitive. The conclusion is precisely

static marginal compatibility ≠ operational strategy characterization.

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midrule En03-A, v1 Three measurement interfaces, effect pullback, residual-instrument distinction, and prescribed-schedule locality. None is a new Born axiom or a solution of P1–P3.

addlinespace En02-B, v1 Resource-certified decoder soundness under fixed Born semantics and a schedule-dependent coherent/adaptive peak separation.

addlinespace Psy02-B, v2 Complete process equality implies equality of permitted adaptive record laws. No converse is attributed to its process theorem.

addlinespace Hawk01-A, v1; Hawk02-A, v1 Finite-channel nonidentifiability from matched marginal and thermal diagnostics; not a microscopic dynamics premise.

addlinespace Hawk01-B, v1 Destructive number-record access is not coherent code recovery.

addlinespace Em03-C, v1; Em03-E, v1 Finite observation and transfer interfaces do not by themselves construct an interior algebra or resolve horizon physics.

addlinespace XIV-I, v1 A named internal amplitude construction is distinct from the unconstructed joint characteristic map and an admitted coefficient.

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## References

N. Cheung. The Enter Calculus: A Linear Quantum Lambda Calculus with an Operational Characterization of Measurement. *Frontiers in Physics*, volume 14, 2026. doi: \nolinkurl{10.3389/fphy.2026.1954943}. Accepted-article record; accepted 28 September 2026. The publisher listed the final formatted version as forthcoming at the bibliographic check on 10 October 2026. https://www.frontiersin.org/journals/physics/articles/10.3389/fphy.2026.1954943/abstract.

P. Selinger and B. Valiron. A lambda calculus for quantum computation with classical control. *Mathematical Structures in Computer Science*, 16(3):527–552, 2006. doi: \nolinkurl{10.1017/S0960129506005238}.

A. M. Gleason. Measures on the closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6(4):885–893, 1957. doi: \nolinkurl{10.1512/iumj.1957.6.56050}.

W. H. Zurek. Environment-assisted invariance, entanglement, and probabilities in quantum physics. *Physical Review Letters*, 90:120404, 2003. doi: \nolinkurl{10.1103/PhysRevLett.90.120404}.

M. Schlosshauer and A. Fine. On Zurek's derivation of the Born rule. *Foundations of Physics*, 35:197–213, 2005. doi: \nolinkurl{10.1007/s10701-004-1941-6}.

K. A. Kirkpatrick. Translation of G. Lüders' ``Über die Zustandsänderung durch den Me{ ss }proze{ ss }''. *Annalen der Physik*, 15(9):663–670, 2006. English translation and discussion of Lüders' original projective state-change paper. doi: \nolinkurl{10.1002/andp.200610207}. arXiv: quant-ph/0403007.

W. F. Stinespring. Positive functions on *C^{*}*-algebras. *Proceedings of the American Mathematical Society*, 6(2):211–216, 1955. doi: \nolinkurl{10.1090/S0002-9939-1955-0069403-4}.

M.-D. Choi. Completely positive linear maps on complex matrices. *Linear Algebra and its Applications*, 10(3):285–290, 1975. doi: \nolinkurl{10.1016/0024-3795(75)90075-0}.

W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299:802–803, 1982. doi: \nolinkurl{10.1038/299802a0}.

D. Frauchiger and R. Renner. Quantum theory cannot consistently describe the use of itself. *Nature Communications*, 9:3711, 2018. doi: \nolinkurl{10.1038/s41467-018-05739-8}.

A. Almheiri, D. Marolf, J. Polchinski, and J. Sully. Black holes: complementarity or firewalls? *Journal of High Energy Physics*, 2013(02):062, 2013. doi: \nolinkurl{10.1007/JHEP02(2013)062}.

N. Cheung. *Enter Calculus Companion*. Registered-manuscript index and source-record interface. https://enter.grok.me/. Registry consulted 10 October 2026. Individual manuscript versions cited below are unpublished records; the supplied excerpts are not independently machine-verified.

N. Cheung. *En03-A*, En03-A.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source locators: def:enter, def:recording, def:nondestructive, thm:pullback, thm:obstruction, and thm:locality. https://enter.grok.me/.

N. Cheung. *En02-B*, En02-B-v1.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source locators: thm:B1, def:moduleclass, and cor:separation. https://enter.grok.me/.

N. Cheung. *Psy02-B*, Psy02-B-v2.tex, version 2. Undated registered manuscript, Enter Calculus Companion. Source locator: thm:process, complete-process equality and testing obstruction. https://enter.grok.me/.

N. Cheung. *Hawk01-A*, Hawk01-A.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source locator: thm:completion, conserving completions with identical late radiation. https://enter.grok.me/.

N. Cheung. *Hawk01-B*, Hawk01-B.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source locator: thm:number, exact number-record recovery obstruction. https://enter.grok.me/.

N. Cheung. *Hawk02-A*, Hawk02-A.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source locators: cor:spectrum and the conditional physical acceptance interface ass:physical. https://enter.grok.me/.

N. Cheung. *Em03-C*, Em03-C-v1.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source section: ``Open physical bridges: not additional release claims'', including the exterior/interior and horizon-smoothness boundaries. https://enter.grok.me/.

N. Cheung. *Em03-E*, Em03-E-v1.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source locator: thm:recovery, exact recovery analysis for its declared finite transfer model. https://enter.grok.me/.

N. Cheung. *XIV-I*, XIV-I.tex, version 1. Undated registered manuscript, Enter Calculus Companion. Source locators: thm:I2, thm:I3, and thm:release. The named internal amplitude construction and the unconstructed joint characteristic map are distinct source claims. https://enter.grok.me/.